IRRATIONAL NUMBERS

AND THEIR REPRESENTATION BY

SEQUENCES AND SERIES

BY

HENRY PARKER MANNING, Ph.D.

ASSISTANT PROFESSOR OF PURE MATHEMATICS
IN BROWN UNIVERSITY

FIRST EDITION

FIRST THOUSAND

NEW YORK
JOHN WILEY & SONS
London: CHAPMAN & HALL, Limited
1906

ROBERT DRUMMOND, PRINTER, NEW YORK

PREFACE

THIS book is intended to explain the nature of irrational numbers, and those parts of Algebra which depend on what is usually called The Theory of Limits.

Many of our text-books define irrational numbers by means of sequences; but to the author it has seemed more natural to define a number, or at least to consider a number as determined, by the place which it occupies among rational numbers, and to assume that a separation of all rational numbers into two classes, those of one class less than those of the other, always determines a number which occupies the point of separation. Thus we have the definition of Dedekind, which is adopted by Weber in his Algebra. Without attempting to inquire too minutely into the significance of this definition, we have endeavored to show how the fundamental operations are to be performed in the case of irrational numbers and to define the irrational exponent and the logarithm.

Defining the irrational number by the place which it occupies among rational numbers, we proceed to speak of its *representation* by sequences; and when we have proved that a sequence which represents a number is regular and that a sequence which is regular repre-

sents a number, we are in complete possession of the theory of sequences and their relation to numbers.

The representation of a number by a sequence is essentially the same as its representation as the limit of a variable, and the notion of sequence seems to be simpler than that of variable and limit. A section has been added on Limits (Chap. II, IV) to bring out the relation of the two points of view. But this section may be omitted without any break in the continuity of the book, and the words "variable" and "limit" are used nowhere else.

The infinite series is defined as a sequence written in a particular way. Theorems are given on the convergency and use of infinite series sufficient to develop the exponential, binomial, and logarithmic series.

The theory of irrational numbers given in Chapter I has been adopted by Professor Fine, as stated elsewhere (p. 56). Perhaps I may be permitted to add that I did not see Professor Fine's book until after my manuscript was in the hands of the printer.

In addition to the references on page 56, mention may be made of two important articles on "The Continuum as a Type of Order," by Dr. E. V. Huntington, in the *Annals of Mathematics* for July and October, 1905, and "Introduction to the Real Infinitesimal Analysis of One Variable," by Professors Oswald Veblen and N. J. Lennes (John Wiley & Sons).

I am indebted to Mrs. Elsie Straffin Bronson, A.M., of Providence, for many criticisms and suggestions.

HENRY P. MANNING.

PROVIDENCE, February, 1906.

CONTENTS

CHAPTER I

Irrational Numbers

CHAPTER II

Sequences

CHAPTER III

Series

CHAPTER IV

Power Series

CHAPTER V

The Exponential, Binomial, and Logarithmic Series

IRRATIONAL NUMBERS

CHAPTER I

IRRATIONAL NUMBERS

I. Infinite Sets of Objects

1. The numbers used in counting,

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots,$$

are called positive integers or natural numbers.

When these numbers are arranged in the order of counting each number is followed by one which comes next after it, and we assume that there is no number which comes last.

When we have a set of objects and some positive integer n is the number of these objects, we say that there is a finite number of objects in the set.

When no integer n is the number of all the objects of a set, the set is said to consist of an infinite number of objects.

We must remember that every positive integer is a finite number. It will be best for us to regard the phrase "infinite number" as meaning simply without number.

With reference to a given set of objects the phrase is used as defined above, to express the fact that any integer n is the number of only a part of the objects of the set.

Each integer except 1 is preceded by a finite number of others in the order of counting, and each integer is followed by an infinite number of others.

2. When we have two infinite sets of objects we can often make them correspond; that is, we can pair them, associating with each object of one set one and only one of the other set.

Thus the even numbers can be paired with the odd numbers.

The integers which are the squares of integers can be paired with those which are not squares, although the latter occur more frequently in counting. In this way we have pairs of numbers as follows:

1 and 2, 4 and 3, 9 and 5, 16 and 6, etc.

In this arrangement every number of either kind is associated with one and only one of the other kind. There is no place where the set of squares is exhausted and the numbers which are not squares have to stand alone.*

In Geometry the points of two circles may be associated in this way, each point of one being associated with one and only one of the other. We can do this,

---

* This illustration was used by Galileo. "Galileo and the Modern Concept of Infinity," Dr. Edward Kasner, *Bulletin of the American Mathematical Society*, June, 1905, p. 499.

for example, by making the centres of the two circles coincide and associating points which lie on the same radius.

We may have a correspondence between two infinite sets of objects when one set is a part of the other set. Thus we can associate the set of even positive integers with the set of all positive integers. In Geometry we can associate the points on two segments of straight lines even when one segment is longer than the other. We can do this by making the two segments two sides of a triangle and associating the points in which they intersect any line parallel to the third side.

3. A rational number is any number which is a positive or negative integer or fraction, or zero.

We shall assume that we know how to add or multiply any two rational numbers, to subtract any rational number from any other or the same rational number, and to divide any rational number by any other or the same rational number, with the single exception that we cannot divide any number by zero.

The result of any of these operations will be a rational number.

Between any two rational numbers there are others. We get one such number by adding to the smaller some part of their difference. Between this and each of the other two another can be found, and so on.

Between any two rational numbers, therefore, there is an infinite number of rational numbers.

If a is any positive rational number, there is an integer n which is greater than a. For, if a is a positive integer any integer that comes after a in counting

will be greater than a, and if $a=\frac{p}{q}$, where p and q are positive integers, any integer greater than p will be greater.

If a and b are any two positive rational numbers, there is an integer n such that $nb>a$;* namely, any integer greater than the rational number $\frac{a}{b}$.

If any particular integer n satisfies either of these conditions, every integer beyond will satisfy the same condition.

4. If we suppose all rational numbers arranged in order of magnitude, then no one is followed by another which comes next after it. For any two of them are separated by others.

We cannot realize by our imagination this arrangement of rational numbers. We can only reason about it. Thus we can say of any two numbers in this arrangement that one comes before the other, or of any three that one comes between the other two.

It is possible, however, to arrange the set of all positive rational numbers so that one of these numbers comes first and each of the others is preceded by only a finite number of them, each one having a definite numbered position. That is, it is possible to make this set of numbers correspond to the set of positive integers, one number of each set to be associated with one and only one number of the other set.

One way of doing this is to arrange these numbers

* This is called the Law of Archimedes.

in the order of magnitude of the sum of the numerator and denominator, those in which the sum is the same being arranged in their own order of magnitude and integers being regarded as fractions with 1 for denominator. This arrangement will be

$$1 \quad \frac{1}{2} \quad 2 \quad \frac{1}{3} \quad 3 \quad \frac{1}{4} \quad \frac{2}{3} \quad \frac{3}{2} \quad 4 \quad \frac{1}{5} \quad 5 \quad \frac{1}{6} \quad \frac{2}{5} \quad \frac{3}{4} \quad \frac{4}{3} \quad \frac{5}{2} \quad 6 \ldots.$$

II. Definition of Irrational Numbers

5. Any rational number a separates all others into two classes, those which are smaller being in one class and those which are larger in the other, and every number of the first class is less than every number of the second class. We may put the number itself into one of the two classes and then we have separated all rational numbers into two classes, every number of the first class less than every number of the second class.

If we have put a into the first class, it is the largest number in this class. In this case there is no number in the second class which is the smallest number in the second class. For any number b in the second class is larger than a, and there are rational numbers between a and b. These numbers are in the second class because larger than a, and they are smaller than b, so that b is not the smallest number in the second class.

If we have put a into the second class, it is the smallest number in the second class and there is no number in the first class which is the largest number in the first class.

In either case a occupies the point of separation of the two classes, and we may think of a as determined by the separation.

Now there are ways in which we can separate all rational numbers into two classes, those of the first class less than those of the second, with no rational number occupying the point of separation, that is, with no number in the first class which is the largest number in the first class, and no number in the second class which is the smallest number in the second class.

For example, there is no rational number whose square is 2.* If we separate all rational numbers into two classes, putting into the first class all negative rational numbers and all positive rational numbers whose squares are less than 2, and into the second class all positive rational numbers whose squares are greater than 2, the numbers in the first class will be less than those in the second class, and there is no rational number which will be the largest number in the first class or the smallest number in the second class.

For, let a be any positive number in the first class, that is, any positive rational number whose square is less than 2. Let p be any other positive rational number. $2-a^2$ is positive, and $(a+p)^2$ or $a^2+p(2a+p)$ will be less than 2 if

$$p(2a+p)<2-a^2.$$

* This is a proposition of Euclid (Elements, X, 117).

This will be true if p is less than some positive number p', and at the same time less than

$$\frac{2-a^2}{2a+p'}.$$

Suppose we take p equal to a half of the smaller of the two numbers

$$1 \quad \text{and} \quad \frac{2-a^2}{2a+1},$$

$a+p$ will be a rational number greater than a and its square will be less than 2.

Again, let a' be any number in the second class and p some other positive rational number less than a'. a'^2-2 is positive, and $(a'-p)^2$ will be greater than 2 if

$$p(2a'-p) < a'^2-2.$$

This will be true if

$$p < \frac{a'^2-2}{2a'}.$$

If we take p equal to a half of this fraction, $a'-p$ will be a positive rational number less than a' and its square will be greater than 2.

We assume:

In any separation of all rational numbers into two classes, those of the first class less than those of the second class, there is a number which occupies the point of separation.

This is the definition of continuity. We assume that the system of all numbers is continuous.

The number is determined by the separation. A number thus determined, if not a rational number, is called an irrational number.

6. An irrational number separating all rational numbers into two classes is to be regarded as greater than those of the first class and less than those of the second class.

Two irrational numbers are different if they do not separate rational numbers into the same two classes. There are numbers which are in the second class in one case and in the first class in the other case. They are greater than one and less than the other of the two irrational numbers.

If a rational number c lies between two irrational numbers, that irrational number which is less than c is to be regarded as the smaller of the two and the other as the larger of the two.

We have, then, the following theorem:

Theorem.—*α, β, and γ being any three numbers, rational or irrational, if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

Proof.—Any rational number c lying between α and β is less than a rational number c' between β and γ, c being in one of the two classes which determine β and c' in the other. But either of these numbers is greater than α and less than γ. Therefore, $\alpha < \gamma$.

In any separation of numbers into two classes determining a number, rational or irrational, we may include

all irrational numbers in the two classes. An irrational number less than a rational number of the first class belongs in the first class, and an irrational number greater than a rational number of the second class belongs in the second class. The number which occupies the point of separation may be put into either class. If it is in the first class, it will be the largest number in this class and there will be no smallest number in the second class, rational or irrational. If it is in the second class, it will be the smallest number in the second class and there will be no largest number in the first class.

7. In Geometry any line commensurable with a given line a has a ratio to a which is a rational number, and any positive rational number is the ratio of some line to a. If the rational number is $\frac{m}{n}$, where m and n are positive integers, we have but to divide a into n equal parts and lay off one of these parts m times on another line to get a line whose ratio to a is $\frac{m}{n}$.

If we take a for unit of length, the ratio of the other line to a is called the length of the other line.

If a line b is not commensurable with a, we may divide all lines which are commensurable with a into two classes, putting those which are shorter than b in the first class and those which are longer in the second. All rational numbers are thus separated into two classes and an irrational number is determined which we call the ratio of b to a.

We may think of all lines as laid off on some in-

definite line L from some fixed point O on L. a being the unit line, the distance of the end-point of any line from O will be the ratio of this line to a. We may let points to the right of O have a positive distance and points to the left of O a negative distance. The distance of O itself will be zero. Every point, then, on the line L will have a distance from O which is a rational or an irrational number.

Now we assume in Geometry:

In any separation of the points of a line into two classes, those of the first class being to the left of those of the second class, there is a definite point of separation, a point whose distance from a given point O on the line, with respect to a given unit of length, is either a rational or an irrational number.

That is, we assume that the points of the line form a continuous system.

Making this assumption, we have the following theorem:

Theorem.—*We can establish a correspondence (Art. 2) between the points of a line and the set of all rational and irrational numbers, associating each point with one and only one number, and with each number one and only one point.*

Proof.—Taking a point O on the line and a unit of length, we have proved that every point on the line has a distance from O which is a rational or an irrational number, and that every rational number is the distance from O of some point on the line.

Any irrational number separating all numbers into two classes determines a separation of the points of

the line into two classes, and so determines a point whose distance from O is this irrational number. Therefore, every irrational number, also, is the distance from O of some point on the line.

If, then, we associate each point of the line with the number which expresses its distance from O, we have a correspondence between the points and the system of all rational and irrational numbers.

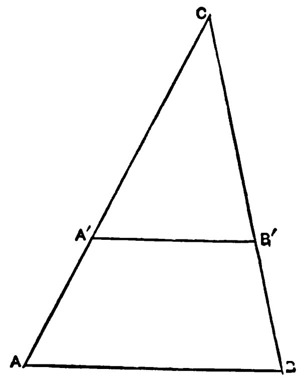
Every line has a ratio, rational or irrational, to a given line a, and every rational or irrational number is the ratio of some line to a.

We make the same assumption of angles, of arcs on the same or equal circles, and of many other magnitudes of Geometry.

8. There are several theorems in which geometrical magnitudes are associated in such a way that any two magnitudes of one set are in the same ratio as the corresponding magnitudes of the other set.

Such a theorem is first proved when the two magnitudes of each set are commensurable. Then, when the two magnitudes of each set are incommensurable, their incommensurable ratios divide all rational numbers into the same two classes, and are, therefore, the same irrational number.

For example, assume it to have been proved that when a line parallel to one side of a triangle cuts off on the other two sides parts which are commensurable with the whole sides, these parts have the same ratio to the whole sides. When the line cuts off parts which are not commensurable with the whole sides, these parts also have the same ratio to the whole sides.



Let $A'B'$ be parallel to the side AB of the triangle ABC and cut off on CA and CB parts CA' and CB' not commensurable with CA and CB, to prove that the incommensurable ratios of CA' to CA and of CB' to CB are the same irrational number.

Any positive rational number which is less than the ratio of CA' to CA is the ratio to CA of a commensurable line shorter than CA', and a parallel to AB cutting off this line, since it cannot cross the parallel $A'B'$, must cut off on CB a line shorter than CB'. But the line so cut off has for ratio to CB the same rational number. Therefore, any positive rational number less than the ratio of CA' to CA, and so any rational number less than the ratio of CA' to CA, is less than the ratio of CB' to CB. In the same way we prove that any rational number less than the ratio of CB' to CB is less than the ratio of CA' to CA.

That is, these two ratios divide all rational numbers into the same two classes, and are, therefore, the same irrational number.

9. Theorem.—*Given a separation of all rational numbers into two classes, those of the first class less than those of the second, and any positive rational number* ε, *there are two numbers, one in each class, whose difference is less than* ε.

Proof.—Let k and k' be two numbers in the first and second class, respectively. There is a positive integer n greater than $\frac{k'-k}{\varepsilon}$ (Art. 3). If we divide $k'-k$ into n equal parts, each of these parts will be less than ε. Let p be one of these parts and add it to k n times in succession. We get the numbers

$$k \quad k+p \quad k+2p \; . \; . \; . \; k'.$$

Some of these numbers will be in the first class and the rest in the second class. Let a be the last one in the first class and let a' be the number which comes next after a in this set of numbers, and which is, therefore, in the second class. Then

$$a'-a=p<\varepsilon.$$

There are, indeed, an infinite number of numbers in each class, any one of those in one class differing from any one of those in the other class by less than ε.

10. Theorem.—*Given a set of rational numbers separated into two classes, those of the first class less than those of the second, if for every positive rational number* ε *there are two numbers, one in each class, whose difference is less than* ε, *we can by means of this separation determine a separation of all rational numbers, and a*

number, rational or irrational, which occupies the point of this separation.

Proof.—To determine a separation of all rational numbers that will include the given separation we must put any rational number into the first class if it is less than a number already in the first class, and into the second class if it is greater than a number already in the second class. There can be only one rational number whose position is not in this way determined. For the difference between any two different rational numbers is a positive number, a value of ε, and there are two numbers in the given set, one in each class, whose difference is less than this value of ε.

If there is one rational number which is not less than any number already in the first class nor greater than any number already in the second class, we may put it into either class. Every rational number is, then, assigned to one or the other class, and a number, rational or irrational, is determined by this separation.

Any separation of numbers into two classes, those of the first class less than those of the second, is called a *cut*. We shall, however, use the phrase *separation into two classes.*

III. Operations upon Irrational Numbers

11. Addition.—If, a, b, c, and d are rational numbers, and if $a < b$ and $c < d$, then $a + c < b + d$.

Now let α and β be two numbers, one or both irrational. Let a be any number in the first and a' any

number in the second of two classes of rational numbers determining the number α. Let b and b' in the same way denote numbers of two classes determining β.

Theorem.—*We can determine a separation of all numbers into two classes by putting numbers of the form $a+b$ into the first class and numbers of the form $a'+b'$ into the second class.*

Proof. $$a+b < a'+b'.$$

Now, ε being any given positive rational number, there are numbers a and a' whose difference is less than $\frac{\varepsilon}{2}$, and numbers b and b' whose difference is less than $\frac{\varepsilon}{2}$. Thus there are numbers $a+b$ and $a'+b'$ whose difference is less than ε, and a separation of numbers is determined.

The number determined by this separation is called the sum of α and β and is written $\alpha+\beta$.

Cor. I.—c being a rational number and α irrational, we can determine the sum $\alpha+c$ by the separation of numbers of the form $a+c$ and $a'+c$.

Proof.—The separation of the numbers $a+c$ and $a'+c$ is sufficient to determine a number, since for any given positive rational number ε we can determine numbers a and a' so that

$$(a'+c)-(a+c) < \varepsilon.$$

Now, given any number a, if k is the difference between a and some rational number lying between a and α, then

$$a+c=(a+k)+(c-k),$$

and by definition $(a+k)+(c-k)$ is a number in the first of two classes determining the sum $\alpha+c$. Similarly we prove that $a'+c$ is a number in the second of two classes determining $\alpha+c$.

The number determined by the separation of the rational numbers $a+c$ and $a'+c$ must, therefore, be the same as the sum $\alpha+c$.

Cor. 2.—The commutative and associative laws hold in the addition of irrational numbers, since they hold in the addition of the rational numbers used in defining the sum of irrational numbers.

That is,

$$\alpha+\beta=\beta+\alpha,$$

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma).$$

12. Theorem.—*α, β, and γ being any three numbers, rational or irrational, if $\alpha<\beta$, then $\alpha+\gamma<\beta+\gamma$.*

Proof.—Take a' and b so that $a'<b$, and let c and c' be rational numbers of two classes determining γ, c and c' taken so that $c'-c<b-a'$. Then we have in succession the inequalities

$$\alpha+\gamma<a'+c'<b+c<\beta+\gamma;$$

whence $\alpha+\gamma<\beta+\gamma.$ **(Art. 6)**

Cor.—If also $\gamma < \delta$, then $\alpha + \gamma < \beta + \delta$.

13. Subtraction.—We can define $\alpha - \beta$ as the number which added to β will give α for sum. For all numbers may be separated into two classes, those which added to β give a sum less than α being in the first class and those which give a sum not less than α in the second class.

Theorem.—*We can separate all numbers into two classes by putting numbers of the form $a - b'$ into the first class and numbers of the form $a' - b$ into the second class, and this separation will determine the number $\alpha - \beta$.*

Proof.— $$a - b' < a' - b,$$

and as in the proof of the theorem of Art. 11 we can determine the numbers a, a', b, and b' so that the difference of the numbers $a - b'$ and $a' - b$ shall be less than ϵ, this difference being the same as the difference of the numbers $a + b$ and $a' + b'$.

Now, given any a and b', $a - b' + b'$ is a number in the second of two classes determining the sum of the rational number $a - b'$ and β (Art. 11, Cor. 1). But, being equal to a, this number is in the first of two classes determining α, and the sum of the rational number $a - b'$ and β is a number less than α.

Similarly we prove that all numbers of the form $a' - b$ added to β give a sum which is greater than α.

The numbers $a - b'$ and $a' - b$, therefore, determine by their separation the number which added to β gives the sum α; they determine the number $\alpha - \beta$.

14. If a and b are any two rational numbers, and if $a<b$, then $-b<-a$.

If we have all rational numbers separated into two classes and we change the signs of all these numbers, we shall get a new separation of all rational numbers, those in either class in the first separation becoming those in the other class in the second separation. Unless these separations determine the number zero they will determine two different numbers, one greater than zero, or positive, and the other less than zero, or negative. These two numbers are said to be numerically equal with opposite signs. Each is the negative of the other, and that one which is positive is the numerical value of both.

The numerical value of a number is also called its modulus, and is often denoted by placing the number between vertical lines. Thus, we write $|\alpha|$ for the numerical value of α.

α and β being any two numbers,

$$\alpha-\beta=\alpha+(-\beta);$$

for both members of this equation are determined by a separation of rational numbers of the form

$$a-b' \quad \text{and} \quad a'-b,$$

which can also be written

$$a+(-b') \quad \text{and} \quad a'+(-b).$$

In a similar way we prove that the numbers $\alpha-\beta$

and $\beta-\alpha$ are numerically equal with opposite signs, so that we may write

$$\beta-\alpha=-(\alpha-\beta),$$

$$|\beta-\alpha|=|\alpha-\beta|.$$

$|\alpha-\beta|$ is called the difference of the two numbers α and β.

The difference of two numbers may always be obtained by changing the sign of one of them and adding, and since the numerical value of the sum of two numbers is equal to the sum or difference of their numerical values, we can say that the numerical value of the sum or difference of two numbers is equal to or less than the sum and equal to or greater than the difference of their numerical values.

That is, we may write

$$|\alpha\pm\beta| \stackrel{=}{<} |\alpha|+|\beta|,$$

and

$$\stackrel{=}{>} |\alpha|-|\beta|,$$

and either letter in the last expression may be written in the first term.

In particular, if $|x-c|<\alpha$ and $|y-c|<\beta$, then

$$|x-y|<\alpha+\beta.$$

15. Multiplication.—If a, b, c, and d are positive rational numbers, $a<b$ and $c<d$, then $ac<bd$.

Now let α and β be two positive numbers, one or both irrational. Let a be any positive number in the first and a' any number in the second of two classes

of rational numbers determining α, and let b and b' represent in the same way two classes determining β.

Theorem.—*We can determine a separation of all numbers into two classes by putting numbers of the form ab into the first class and numbers of the form $a'b'$ into the second class.*

Proof. $$ab < a'b'.$$

Now let ϵ be any positive rational number. p being any other positive rational number, there are numbers a and a' whose difference is less than p, and numbers b and b' whose difference is less than p. That is, there are numbers for which

$$a' < a + p \quad \text{and} \quad b' < b + p;$$

therefore, $$a'b' < ab + p(a + b + p).$$

If p is taken less than some particular positive rational number p_1 and we let a_1 and b_1 be two particular rational numbers greater than α and β, respectively, so that

$$a + b + p < a_1 + b_1 + p_1,$$

and if we also take p less than

$$\frac{\epsilon}{a_1 + b_1 + p_1},$$

we shall have $$a'b' < ab + \epsilon,$$

or $$a'b' - ab < \epsilon.$$

The number determined by this separation is called the product of α and β and is written $\alpha\beta$.

If one of the two numbers is zero, we shall say that the product is zero. If one of the numbers is negative, or if both are negative, the numerical value of the product will be the product of their numerical values, and the product itself will be positive or negative according as they have like or unlike signs.

Cor. 1.—c being a rational number and α irrational, we can determine the product αc by the separation of numbers of the form ac and $a'c$.

Proof.—The separation of the numbers ac and $a'c$ is sufficient to determine a number, since we can make

$$a'c - ac < \varepsilon$$

by taking a and a' so that

$$a' - a < \frac{\varepsilon}{c}.$$

Now, given any a, if k is the ratio to a of some rational number lying between a and α, so that the number between a and α may be written ak, then

$$ac = ak \cdot \frac{c}{k},$$

and by definition this is a number in the first of two classes determining the product αc.

Similarly we prove that $a'c$ is a number in the second of two classes determining αc.

Therefore, the number determined by the separation of the numbers ac and $a'c$ must be the same as the number αc.

Cor. 2.—The commutative, distributive, and associative laws hold in the multiplication of irrational numbers, since they hold in the multiplication of the rational numbers used in defining the product of irrational numbers.

That is,
$$\alpha\beta=\beta\alpha,$$
$$(\alpha\beta)\gamma=\alpha(\beta\gamma),$$
$$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma.$$

16. Theorem.—*α, β, and γ being any three positive numbers, rational or irrational, if $\alpha<\beta$, then $\alpha\gamma<\beta\gamma$.*

Proof.—Take a' and b so that $a'<b$; let c_1 be some particular rational number less than γ and take c and c' rational numbers of two classes determining γ, c and c' taken so that

$$c>c_1 \text{ and at the same time } c'-c<\frac{c_1(b-a')}{a'}.$$

We shall then have

$$c'-c<\frac{c(b-a')}{a'},$$

or
$$a'c'<bc.$$

That is,
$$\alpha\gamma<a'c'<bc<\beta\gamma.$$

Cor.—If also $\gamma<\delta$, then $\alpha\gamma<\beta\delta$.

17. Division.—α and β being any two numbers, one or both irrational and α not zero, we can define the quotient $\frac{\beta}{\alpha}$ as the number which multiplied by α will give β for product.

For, if α and β are positive, we can separate all numbers into two classes by putting into the first class those numbers which multiplied by α give a product less than β, and into the second class those which multiplied by α give a product not less than β. This separation determines a number which multiplied by α gives β for product.

If β is zero, the quotient will be zero.

If one number is negative, or if both numbers are negative, the numerical value of the quotient will be the quotient of their numerical values, and the quotient itself will be positive or negative according as they have like or unlike signs.

α and β being positive, let k be some positive rational number less than α, and for the moment restrict a to rational numbers which are greater than k in the first of the two classes determining α. a', b, and b' being defined as before, we have the following theorem:

Theorem.—*We can separate all numbers into two classes by putting numbers of the form* $\frac{b}{a'}$ *into the first class and numbers of the form* $\frac{b'}{a}$ *into the second class, and this separation will determine the number* $\frac{\beta}{\alpha}$.

Proof. $$\frac{b}{a'} < \frac{b'}{a},$$

and $$\frac{b'}{a} - \frac{b}{a'} < \frac{a'b' - ab}{k^2}.$$

Now as in the proof of the theorem for multiplication we have

$$a'b' - ab < p(a_1 + b_1 + p_1),$$

and if we take p also less than

$$\frac{\varepsilon k^2}{a_1 + b_1 + p_1},$$

we shall have $$\frac{b'}{a} - \frac{b}{a'} < \varepsilon.$$

But, given any a' and b, $\frac{b}{a'}a'$ is a number in the second of two classes determining the product of the rational number $\frac{b}{a'}$ and α (Art. 15, Cor. 1); and, being equal to b, this number is in the first of two classes determining β. Thus the product of the rational number $\frac{b}{a'}$ and α is a number less than β.

Similarly we prove that all numbers of the form $\frac{b'}{a}$ multiplied by α give a product which is greater than β.

The numbers $\frac{b}{a'}$ and $\frac{b'}{a}$, therefore, determine by their

separation the number which multiplied by α gives the product β: they determine the number $\frac{\beta}{\alpha}$.

IV. Exponents and Logarithms

18. We obtained an example of an irrational number by separating all rational numbers into two classes, those of the first class comprising all negative rational numbers and all positive rational numbers whose squares are less than 2, those of the second class comprising all positive rational numbers whose squares are greater than 2. We can now say that the irrational number so determined is a number whose square is equal to 2, and that it may be called the square root of 2.

In general we have the following theorem:

Theorem.—*α being any positive rational or irrational number and n any positive integer, there exists a positive number β whose nth power equals α.*

Proof.—We can separate all positive numbers into two classes according as their nth powers are less than α or not less than α. Let a be any positive number whose nth power is less than α, and b any positive number whose nth power is not less than α. Then a must be less than b; for, otherwise, the nth power of a would not be less than the nth power of b (Art. 16). Putting negative numbers into the first class we have, therefore, a separation in which all numbers of the

first class are less than those of the second class, and a number β is determined by this separation.

Now α can be determined by the separation into two classes of numbers of the form a^n and b^n. Therefore, the nth power of β is equal to α.

β is called the nth root of α and is written $\sqrt[n]{\alpha}$.

19. Theorem.—*If a number α is greater than 1, and ϵ is any positive number, there is an integer n such that*

$$\alpha^n > \epsilon.$$

Proof.—Let $\alpha = 1 + \delta.$

Then we can prove by induction

$$\alpha^n \geqq 1 + n\delta.$$

For, assuming that this is true for n, we have

$$\alpha^{n+1} \geqq (1 + n\delta)(1 + \delta)$$

$$> 1 + (n+1)\delta,$$

so that what we have assumed, if true for n, is true for $n+1$, and, being true for $n=1$, is true for all values of n.

Therefore, for all values of n

$$\alpha^n > n\delta.$$

Now by Art. 3 there is an integer n for which $n\delta > \epsilon$ (if δ or ϵ is irrational, n may be any integer greater

than some rational number in the second of two classes determining $\left.\frac{\varepsilon}{\delta}\right)$. For any such integer n

$$\alpha^n > \varepsilon.$$

Cor.—If a number α is greater than 1 and ε is any positive number, there is an integer n such that

$$\sqrt[n]{\alpha}-1<\varepsilon.$$

Proof.—Let β be a number lying between 1 and $1+\varepsilon$. There is, then, an integer n such that

$$\beta^n>\alpha,$$

and for this number n

$$\sqrt[n]{\alpha}<\beta<1+\varepsilon;$$

that is, $$\sqrt[n]{\alpha}-1<\varepsilon.$$

$\sqrt[n]{\alpha}$ is greater than 1 and differs from 1 by less than ε.

If a particular integer n satisfies the theorem or the corollary, the same will be true of any integer beyond.

20. Theorem.—*l, r, and s being positive integers and a a positive number,*

$$(\sqrt[ls]{a})^{lr}=(\sqrt[s]{a})^r.$$

Proof.—Let α denote the lsth root of a, so that $\alpha^{ls}=a$.

Then $$\alpha=\sqrt[ls]{a} \quad \text{and} \quad \alpha^{lr}=(\sqrt[ls]{a})^{lr}.$$

But α^{ls} and α^{lr} may be regarded, respectively, as the product of s factors and the product of r factors each equal to α^{l}. Therefore, in the first place,

$$\alpha^{l}=\sqrt[s]{a},$$

and then,
$$\alpha^{lr}=(\sqrt[s]{a})^{r}.$$

That is,
$$(\sqrt[ls]{a})^{lr}=(\sqrt[s]{a})^{r}.$$

Using fractional exponents we write the last theorem

$$a^{\frac{lr}{ls}}=a^{\frac{r}{s}}.$$

That is, if p denotes any positive rational number, we have a definite meaning for the expression a^{p}.

We shall understand that a negative exponent is defined by writing

$$a^{-p}=\frac{1}{a^{p}},$$

and that $a^{0}=1$.

21. Theorem.—*a being a positive number, rational or irrational, and p and q any rational numbers,*

$$a^{p}a^{q}=a^{p+q}.$$

Proof.—First, when p and q are both positive, let s be their common denominator, so that we can write

$$p=\frac{r}{s} \quad\text{and}\quad q=\frac{r'}{s}.$$

Then, if $\alpha^s = a$,

$$a^p = \alpha^r, \quad a^q = \alpha^{r'}, \quad \text{and} \quad a^{p+q} = \alpha^{r+r'}.$$

But $$\alpha^r \alpha^{r'} = \alpha^{r+r'},$$

our theorem for positive integer exponents being a particular case of the associative law. Therefore,

$$a^p a^q = a^{p+q}.$$

Suppose one of these exponents negative and numerically smaller than the other, say $q = -q'$, where q' is positive and less than p. Then

$$a^p a^q = \frac{a^p}{a^{q'}} = a^{p-q'};$$

for $$a^{q'} a^{p-q'} = a^p.$$

If both exponents are negative, or if one exponent is negative and numerically greater than the other, let $p = -p'$ and $q = -q'$.

$$a^p a^q = \frac{1}{a^{p'}} \frac{1}{a^{q'}} = \frac{1}{a^{p'+q'}} = a^{p+q}.$$

The formula reduces to an identity when either exponent is zero.

22. Theorem.—*a being a positive number, rational or irrational, and p and q any rational numbers,* $p < q$,

$$a^p < a^q \quad \textit{when} \quad a > 1,$$

and $$a^p > a^q \quad \text{"} \quad a < 1.$$

Proof.—First, when a is greater than 1.

Using the same notation we have, when p and q are positive,

$$a^p = \alpha^r \quad \text{and} \quad a^q = \alpha^{r'}, \quad r < r'.$$

But when $a > 1$, $\alpha > 1$ and $\alpha^r < \alpha^{r'}$, the latter power being obtained from the former by additional factors α greater than 1. Therefore,

$$a^p < a^q.$$

If p and q are negative, say $p = -p'$ and $q = -q'$, $q' < p'$, then

$$a^{q'} < a^{p'} \quad \text{and} \quad a^p < a^q.$$

If p is negative and q positive,

$$a^p < 1 \quad \text{and} \quad a^q > 1; \quad \text{hence,} \ a^p < a^q.$$

If either exponent is zero, we have

$$1 < a^q \quad \text{or} \quad a^p < 1.$$

Second, when $a < 1$ put $a = \frac{1}{b}$, so that $b > 1$. Then

$$b^p < b^q \quad \text{and} \quad a^p > a^q.$$

Cor.—If p and q have the same sign, $|p| < |q|$, then

$$|a^p - 1| < |a^q - 1|.$$

23. Theorem.—*a being a positive number different from 1, and p and q any rational numbers each numer-*

ically less than some given number M, to every positive number ε corresponds a positive number δ such that

$$|a^q - a^p| < \varepsilon \quad \textit{when} \quad |q-p| < \delta.$$

Proof.—Take $p<q$, say $q-p=r$. Then

$$a^q - a^p = a^p(a^r - 1),$$

and if $a>1$, $\qquad |a^q - a^p| < a^M(a^r - 1).$

Now there is a positive integer n such that

$$a^{\frac{1}{n}} - 1 < \frac{\varepsilon}{a^M}, \qquad \text{(Art. 19, Cor.)}$$

and if we take $\delta = \frac{1}{n}$, we shall have, for values of r less than δ,

$$|a^q - a^p| < \varepsilon. \qquad \text{(Art. 22, Cor.)}$$

If $a<1$, put $a=\frac{1}{b}$ so that $b>1$, and

$$|a^q - a^p| = |b^{-q} - b^{-p}| < \varepsilon$$

when $\qquad |-q+p| < \delta,$

δ being equal to the fraction $\frac{1}{n}$ determined so that

$$b^{\frac{1}{n}} - 1 < \frac{\varepsilon}{b^M}.$$

24. Theorem.—*a being a positive number different from 1, the separation of all rational numbers p into two classes determining an irrational number λ produces a separation of all the numbers a^p, and so of all numbers, into two classes, determining a number, rational or irrational, which we may call a^λ.*

Proof.—Let p_1 be any number in the first and p_2 any number in the second of two classes of rational numbers determining λ. Then

$$a^{p_1} < a^{p_2} \quad \text{when} \quad a > 1,$$

and

$$a^{p_1} > a^{p_2} \quad \text{"} \quad a < 1.$$

Thus in either case the numbers a^p are separated into two classes, those of one class less than those of the other class.

Moreover, there are numbers a^{p_1} and a^{p_2} whose difference is less than any given positive number ϵ, since there are numbers p_1 and p_2 whose difference is less than the number δ of the last theorem.

Therefore, the separation of the numbers a^p into two classes determines a separation of all numbers into two classes, and a number, rational or irrational, which occupies the point of this separation.

The preceding theorems, stated only for rational exponents, can be proved true also of irrational exponents.

For example, let p_1 and p_2 be any two numbers of two classes of rational numbers determining λ, and q_1

and q_2 rational numbers of two classes determining μ, λ and μ being one or both irrational.

The separation of the numbers $a^{p_1}a^{q_1}$ and $a^{p_2}a^{q_2}$ into two classes determines the same number, the product $a^\lambda a^\mu$, as the separation of the same numbers written $a^{p_1+q_1}$ and $a^{p_2+q_2}$ into the same two classes, which determines the number $a^{\lambda+\mu}$. Therefore

$$a^\lambda a^\mu = a^{\lambda+\mu}.$$

25. Theorem.—*a and b being any two positive numbers, a different from 1, there is a number λ such that*

$$a^\lambda = b.$$

Proof.—We can separate all rational numbers into two classes, putting a number p into one class if $a^p < b$, and into the other class if $a^p \geqq b$. By the theorem of Art. 22 the numbers p of one class will be less than those of the other class. This separation determines a number λ, and b is the number denoted by a^λ.

λ is called the logarithm of b to the base a.

CHAPTER II

SEQUENCES

I. Representation of Numbers by Sequences

26. A sequence is an infinite set of numbers arranged so that each one has a definite numbered position; that is, one comes first, each is followed by one that comes next after it, each except the first is preceded by a finite number of others, and each is followed by an infinite number of others.

A sequence is a set of numbers placed in correspondence with the set of positive integers (Art. 2).

The numbers of a sequence are called its elements.

A sequence may be expressed by a formula which gives the nth element for every value of n, or by a statement which indicates in some way how each element is determined.

A sequence is often indicated by a certain number of the elements at the beginning followed by dots, but we ought always to give a formula or to state the law by which the elements are determined. The formula or law need apply, however, only to those numbers not given. Indeed, we may put for any finite number of the elements any numbers we please, and give a formula which applies only to the numbers not written.

We will give the following examples of sequences:

(1) The sequence

$$1 \quad \frac{1}{2} \quad \frac{1}{3} \cdots \frac{1}{n} \cdots.$$

(2) The sequence of prime numbers

$$1 \quad 2 \quad 3 \quad 5 \quad 7 \ldots.$$

(3) The sequence

$$\frac{1}{3} \quad \frac{2}{5} \cdots \frac{n}{2n+1} \cdots.$$

(4) The sequence

$$1 \quad 2 \quad 4 \quad 3 \quad 5 \quad \frac{4}{3} \quad \frac{5}{4} \cdots \frac{n-2}{n-3} \cdots,$$

n any integer greater than 5.

(5) A sequence of fractions whose denominators are the successive integers beginning with 3, the fractions corresponding to even integers all equal to $\frac{1}{2}$ and the fractions corresponding to odd integers alternately less and greater than $\frac{1}{2}$, differing from $\frac{1}{2}$ by the smallest amount possible for fractions with the given denominators; namely, the sequence

$$\frac{1}{3} \quad \frac{2}{4} \quad \frac{3}{5} \quad \frac{3}{6} \quad \frac{3}{7} \quad \frac{4}{8} \quad \frac{5}{9} \cdots.$$

(6) The sequence whose first element is 1, and in

which from any element a we get the next by adding to a the largest integer not greater than $\sqrt{a}$. The first ten elements of this sequence will be

$$1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \quad 10 \quad 13 \quad 16 \quad 20.$$

(7) We may suppose a sequence to consist of one number or certain numbers repeated. For example,

$$2 \quad 2 \quad 2 \ldots 2 \ldots .$$

27. It is possible to arrange a set of numbers in a way that will form a sequence, and in another way that will not form a sequence. We may, for example, select an infinite set from the elements of a sequence and let the remaining elements come after them. Thus, taking the set of positive integers, we might suppose all odd numbers to come first, in order, and then all even numbers. In this arrangement there is a number, 1, which comes first, and each number is followed by one which comes next after it; but any even number is preceded by an infinite number of other numbers. The numbers in this arrangement do not form a sequence.

We have already seen that the set of all positive rational numbers may be arranged so as to form a sequence although these numbers do not form a sequence when arranged in order of magnitude (Art. 4). Indeed, the set of positive integers, and consequently the numbers of any sequence, may be arranged so that between any two there are others,—so that no number of the set is followed by one which comes

next after it. To do this we have but to establish a correspondence between the set of positive integers and the set of all positive rational numbers, as is done on page 5, and then to arrange the set of positive integers in the order of magnitude of the corresponding rational numbers. In such an arrangement of integers each integer is preceded by an infinite number of others, as well as followed by an infinite number of others.

28. An irrational number is determined by the place which it occupies among rational numbers, but it is often most conveniently represented by a sequence.

The sequence

$$a_1 \quad a_2 \; . \; . \; . \; a_n \; . \; . \; .$$

represents a number a if for every positive number ϵ there is a place in the sequence beyond which all the elements differ from a by less than ϵ.

Thus the sequence

$$\frac{1}{2} \quad \frac{2}{3} \; . \; . \; . \; \frac{n}{n+1} \; . \; . \; .$$

represents the number 1.

For, the difference between the number 1 and $\frac{n}{n+1}$, the nth element of this sequence, is $\frac{1}{n+1}$, and this is less than ϵ if $n>\frac{1}{\epsilon}-1$. Now there is a positive

integer n which is greater than $\frac{1}{\epsilon}-1$, and all integers beyond are greater still. Therefore, there is a place in the given sequence beyond which all the elements differ from 1 by less than ϵ.

A sequence cannot represent two different numbers, for no element could differ from both by less than a half of their difference.

If a sequence represents a number a, we may speak of a as the value of the sequence, or say that the sequence is equal to a.

We sometimes say that a sequence is convergent if it represents a number.

29. Theorem.—*A sequence represents the number zero if for every positive number ϵ there is a place in the sequence beyond which all its elements are numerically less than ϵ.*

Proof.—The difference between a number and zero is the numerical value of the number.

Thus the sequence

$$1 \quad \frac{1}{2} \; . \; . \; . \; \frac{1}{n} \; . \; . \; .$$

represents the number zero. For, $\frac{1}{n}<\epsilon$ if $n>\frac{1}{\epsilon}$, and, given any value of ϵ, there is a point beyond which this is true for all values of n.

30. We often use the expression "any given number", "any assigned number", "an arbitrary number", or "any number", meaning every number.

That is, the theorem in which such an expression is used is true for every number.

The expression "any number" may be used with certain restrictions. In fact, we generally restrict it to positive numbers, and we sometimes restrict it to positive rational numbers.

We have made use of expressions of this kind in Arts. 9 and 10 and in Arts. 19 and 23.

We shall generally use the letter ϵ to represent "any positive number".

31. When a sequence represents a number a, the place beyond which all the elements differ from a by less than ϵ will depend, in general, on the value of ϵ.

We may say, however,

(1) If a particular place satisfies the condition for a particular value of ϵ, any place beyond will satisfy the condition for the same value of ϵ.

(2) If a particular place satisfies the condition for a particular value of ϵ, it will satisfy the condition for any larger value of ϵ.

Given two values of ϵ, the first place which satisfies the condition for the smaller value of ϵ will generally, though not always, be farther along than the first place which satisfies the condition for the larger value of ϵ.

There is no place which satisfies the condition for all values of ϵ unless there is a place beyond which the elements of the sequence are all equal to a.

A place in the sequence beyond which all the elements differ from a by less than ϵ may be called a place corresponding to ϵ with respect to a. If a sequence

represents a number a, then to every ϵ corresponds some place in the sequence with respect to a.

If a sequence does not represent a, there must be at least one value of ϵ for which the above condition is not satisfied. By virtue of (2) the same is true of every smaller value of ϵ. Let ϵ_1 be a value of ϵ to which no place corresponds with respect to a. Then there are in the sequence an infinite number of elements which differ from a by as much as ϵ_1. For, beyond some particular place there is at least one, say α_1; beyond α_1 there is at least one, α_2; beyond α_2 another, and so on without end.

32. Theorem.—*If α is a number numerically less than 1, the sequence*

$$\alpha \quad \alpha^2 \; . \; . \; . \; \alpha^n \; . \; . \; .$$

represents the number zero.

Proof.—Since $|\alpha| < 1$, $\dfrac{1}{|\alpha|} > 1$, and for any positive number ϵ there is an integer n such that

$$\left(\frac{1}{|\alpha|}\right)^n > \frac{1}{\epsilon}, \qquad \text{(Art. 19)}$$

or

$$|\alpha|^n < \epsilon.$$

This, being true for a particular integer n, is true for all greater integers.

Again, since $|\alpha|^n$ is the same as $|\alpha^n|$ (Art. 15, just before Cor. 1), there is a positive integer corresponding

to ϵ such that for all integers greater than this integer

$$|\alpha^n| < \epsilon.$$

That is, for any positive number ϵ there is a place in the sequence of powers of α beyond which these powers are numerically less than ϵ.

33. Theorem.—*Any number may be represented by a sequence of rational numbers.*

Proof.—Let

$$\alpha_1 \quad \alpha_2 \; . \; . \; . \; \alpha_n \; . \; . \; .$$

be a sequence of rational numbers representing zero, none of the numbers being themselves zero. To form a sequence representing a number a we may take for a_1 the greatest multiple of α_1 which is not greater than a, for a_2 the greatest multiple of α_2 which is not greater than a, and so on. Then the sequence

$$a_1 \quad a_2 \; . \; . \; . \; a_n \; . \; . \; .$$

will represent the number a. For, a_n differs from a by less than α_n, and for any ϵ there is a place in the sequence of α's beyond which they are all numerically less than ϵ.

Suppose we wish to form a sequence representing the square root of 2 and we take for the α's the sequence

$$1 \quad \frac{1}{2} \quad \frac{1}{3} \; . \; . \; . \; \frac{1}{n} \; . \; . \; .$$

The largest multiple of $\frac{1}{n}$ less than $\sqrt{2}$ will have for numerator the largest integer whose square is less than $2n^2$. The largest squares less than the numbers

$$2 \quad 8 \quad 18 \quad 32 \quad 50 \ldots$$

are

$$1 \quad 4 \quad 16 \quad 25 \quad 49 \ldots$$

Therefore, the square root of 2 is represented by the sequence

$$1 \quad \frac{2}{2} \quad \frac{4}{3} \quad \frac{5}{4} \quad \frac{7}{5} \ldots$$

The diagram opposite will serve to show how these numbers represent the square root of 2.

To represent a rational number we may take a sequence all of whose elements are equal to this number.

On the other hand, we may have sequences whose elements are irrational numbers.

34. Theorem.—*If a sequence represents a number a, the numerical values of its elements form a sequence representing the numerical value of a.*

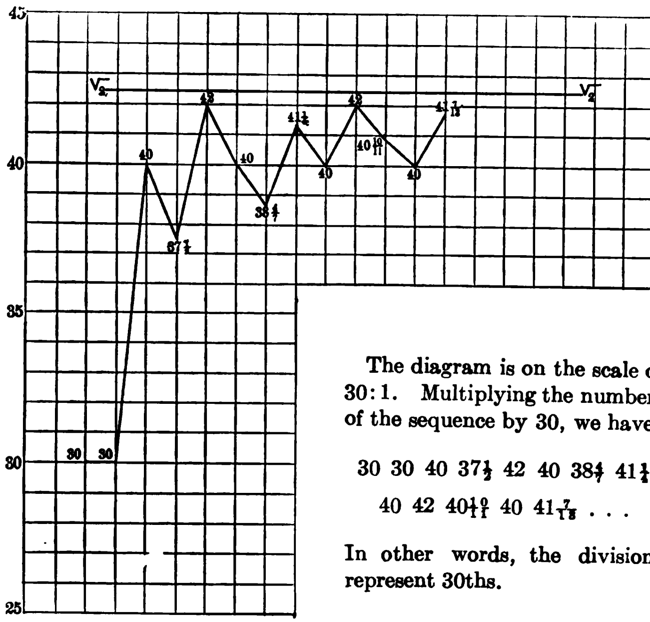
Proof.—If

$$|a_n - a| < \varepsilon,$$

then

$$||a_n| - |a|| < \varepsilon,$$

the latter difference being equal to or less than the former (Art. 14). Therefore, if any place in the origi-



The diagram is on the scale of 30:1. Multiplying the numbers of the sequence by 30, we have

$$30 \quad 30 \quad 40 \quad 37\tfrac{1}{2} \quad 42 \quad 40 \quad 38\tfrac{4}{7} \quad 41\tfrac{1}{4}$$

$$40 \quad 42 \quad 40\tfrac{10}{11} \quad 40 \quad 41\tfrac{7}{13} \quad \ldots$$

In other words, the divisions represent 30ths.

nal sequence corresponds to ϵ with respect to a, the same place in the sequence of numerical values will correspond to ϵ with respect to the numerical value of a.

35. Theorem.—*If a sequence represents a number a, and M is any number different from a, there is a place in the sequence beyond which all the elements differ from a by less than the difference between a and M.*

Proof.—The difference between a and M is a positive number not zero, a particular value of ϵ. The theorem, then, follows from definition.

Cor. 1.—If $M > a$, there is a place beyond which all the elements are less than M.

Cor. 2.—If $M < a$, there is a place beyond which all the elements are greater than M.

Cor. 3.—If a is not zero, there is a place beyond which all the elements have the same sign as a.

For, if a is not zero, we can take zero for M, and any number between zero and a has the same sign as a.

36 Theorem.—*Given two convergent sequences, if there is a place beyond which the elements of the first are greater than the corresponding elements of the second, the number represented by the first is equal to or greater than the number represented by the second.*

Proof.—Let the two sequences be

$$a_1 \quad a_2 \ . \ . \ . \ a_n \ . \ . \ . \ \text{representing } a,$$

and $$b_1 \quad b_2 \ . \ . \ . \ b_n \ . \ . \ . \qquad\qquad\qquad b,$$

and suppose for all values of n

$$a_n > b_n.$$

If $a < b$, take $\varepsilon = \frac{1}{2}(b-a)$. For all values of n beyond a certain place we have

$$a_n < a + \varepsilon \quad \text{and} \quad b_n > b - \varepsilon;$$

and, therefore, $a_n - b_n < a - b + 2\varepsilon = 0$;

that is, $$a_n < b_n,$$

which is contrary to hypothesis. Therefore,

$$a \geqq b.$$

Cor.—If in a convergent sequence there is a place beyond which all the elements are greater than a certain fixed number b, then the number which the sequence represents is equal to or greater than b; or if there is a place beyond which all the elements are less than b, the number which the sequence represents is equal to or less than b.

37. Theorem.—*If each element of a sequence is greater than the preceding and less than some number M, the sequence represents a number, either M or some smaller number.*

Proof.—We can separate all numbers into two classes, putting any number into the first class if there is a place in the sequence beyond which the elements are greater than this number, into the second class if there is no place beyond which the elements are greater than this number. The elements themselves are numbers in the first class, each being less than any that come after it. M is a number in the second class. This separation of all numbers determines a number a, rational or irrational.

Now, ε being any positive number, $a-\varepsilon$ is a number in the first class, and there is a place in the sequence beyond which the elements are all greater than $a-\varepsilon$. But no element is greater than a. Therefore, there is a place beyond which the elements differ from a by less than ε, and hence the sequence represents a.

a belongs to the second class and is the smallest number in the second class. Therefore, a is equal to or less than M.

In the same way we may prove

Theorem.—*If each element of a sequence is less than the preceding and greater than some number M, the sequence represents a number, either M or some larger number.*

II. Regular Sequences

38. The sequence

$$a_1 \quad a_2 \; . \; . \; . \; a_n \; . \; . \; .$$

is regular if for every ε there is a place in the sequence beyond which all the elements differ from one another by less than ε.

39. Theorem.—*If a sequence represents a number a, it is regular.*

Proof.—If a sequence represents a number a, there is a place beyond which all the elements differ from a by less than $\frac{\varepsilon}{2}$, and beyond this place they will differ from one another by less than ε.

40. Theorem.—*Conversely, a regular sequence represents a number.*

Proof.—Given a regular sequence

$$a_1 \quad a_2 \; . \; . \; . \; a_n \; . \; . \; . \; ,$$

we can by means of this sequence determine a separation of numbers as follows:

Let any number c be in the first class if there is a place in the sequence beyond which all the elements are greater than c; otherwise let c be in the second class.

There are numbers in each class. For, there is a place in the sequence beyond which all the elements differ from one another by less than ϵ. a_n being an element beyond this place, $a_n - \epsilon$ is a number in the first class and $a_n + \epsilon$ is a number in the second class.

This separation of numbers into two classes determines a number a, rational or irrational.

Now $a - \epsilon$ belongs to the first class, and there is a place in the sequence beyond which all the elements are greater than $a - \epsilon$.

Again, $a + \frac{\epsilon}{2}$ belongs to the second class, and beyond any place in the sequence there are elements not greater than this number. But there is a place beyond which all the elements differ from each other by less than $\frac{\epsilon}{2}$, and if some of them are not greater than $a + \frac{\epsilon}{2}$, they must all be less than $a + \epsilon$.

We have, then, a place in the sequence beyond which the elements are all greater than $a - \epsilon$, and a place beyond which they are all less than $a + \epsilon$. Beyond the farther of these two places they will all differ from a by less than ϵ.

Therefore, the sequence represents the number a.

III. Operations upon Sequences

41. Theorem.—*If a sequence represents a number a, any sequence formed by taking a part of its elements will represent the same number a.*

Proof.—Each element of the new sequence is at least as far along in the original sequence, and a place which corresponds to ε in the original sequence will certainly correspond to ε in the new sequence.

In this theorem we suppose that the order of any two elements in the new sequence is the same as in the original sequence, that the elements taken form a sequence and are not simply some finite number of elements, and that no element taken is repeated except as it may have been repeated in the original sequence.

The theorem may also be stated as follows:

Theorem.—*If a sequence represents a number a, any sequence which we get by leaving out a part of its elements will represent the same number a.*

Cor. 1. The value of a regular sequence is not changed by interpolating elements if the new sequence is regular.

Cor. 2. If two regular sequences have an infinite number of elements in common, they represent the same number.

42. Theorem.—*If a sequence represents a number a, we may change the order of its elements in any manner, and it will still represent the number a if the elements in*

the new arrangement actually form a sequence, each element having a definite numbered position with only a finite number of elements coming before it (see Art. 27).

Proof.—Let us suppose that all the elements of the original sequence beyond the first n differ from a by less than ϵ. These first n elements will occupy certain definite positions in the second sequence, and beyond the farthest of them the elements of the new sequence will all differ from a by less than ϵ. Therefore, the new sequence represents the same number a.

The number represented by a sequence does not depend on the particular order of the elements. In fact, we may represent a number a by an infinite set of numbers not thought of as arranged in any order whatever, the set being defined so that it is determined of every number that it is a number of the set or that it is not a number of the set.

To do this we should have the following definition:

An infinite set of numbers represents a if for every ϵ it is true that all but a certain number of them differ from a by less than ϵ.

Thus we may dispense with the element of order, but any set of numbers which represents a number a may be arranged as a sequence,* and for us it will be simpler and in many theorems necessary to consider the set as so arranged.

* For example, when an infinite number of them are less than a and an infinite number of them are greater than a, we might take the numbers of the two sets alternately, those less than a in order of magnitude and those greater than a in descending order of magnitude.

43. Theorem.—*If the differences of corresponding elements of two sequences form a sequence equal to zero, and one of the sequences represents a number a, the other sequence represents the number a, and the two sequences are equal.*

Proof.—Let the sequences be

$$a_1 \quad a_2 \quad . \; . \; . \quad a_n \quad . \; . \; . \text{ representing } a,$$

and $$b_1 \quad b_2 \quad . \; . \; . \quad b_n \quad . \; . \; . \; .$$

If there is a place beyond which the b's differ from the corresponding a's by less than $\frac{\varepsilon}{2}$, and a place beyond which the a's differ from a by less than $\frac{\varepsilon}{2}$, then beyond the farther of these two places the b's will differ from a by less than ε. This being true for every ε, the b's must form a sequence representing a.

44. Theorem.—*If two sequences represent two numbers a and b, the sums of their corresponding elements form a sequence representing the sum $a+b$.*

Proof.—There is a place in the first sequence beyond which its elements differ from a by less than $\frac{\varepsilon}{2}$ and a place in the second sequence beyond which its elements differ from b by less than $\frac{\varepsilon}{2}$. Beyond the farther of these two places the sum of corresponding elements

from the two sequences will differ from $a+b$ by less than ε.

45. Theorem.—*If the two sequences*

$$a_1 \quad a_2 \ . \ . \ . \ a_n \ . \ . \ .$$

and

$$b_1 \quad b_2 \ . \ . \ . \ b_n \ . \ . \ .$$

represent the numbers a and b, respectively, the sequence

$$a_1-b_1 \quad a_2-b_2 \ . \ . \ . \ a_n-b_n \ . \ . \ .$$

will represent the number $a-b$.

Proof.—$(a_n-b_n)-(a-b)=(a_n-a)-(b_n-b)$.

Therefore,

$$|(a_n-b_n)-(a-b)| \overline{\overline{<}} |a_n-a|+|b_n-b|. \qquad \text{(Art. 14)}$$

Now there is a place in the first sequence beyond which its elements differ from a by less than $\frac{\varepsilon}{2}$, and a place in the second sequence beyond which its elements differ from b by less than $\frac{\varepsilon}{2}$. That is, beyond the farther of these two places

$$|a_n-a|<\frac{\varepsilon}{2}, \quad |b_n-b|<\frac{\varepsilon}{2},$$

and, therefore, $|(a_n-b_n)-(a-b)|<\varepsilon$.

46. Theorem.—*If two sequences represent two numbers a and b, the products of their corresponding elements will form a sequence representing the product ab.*

Proof.—Let the sequences be

$$a_1 \quad a_2 \ . \ . \ . \ a_n \ . \ . \ . \qquad \text{representing } a,$$

and

$$b_1 \quad b_2 \ . \ . \ . \ b_n \ . \ . \ . \qquad \text{"} \qquad b.$$

We are to prove that the sequence

$$a_1b_1 \quad a_2b_2 \ . \ . \ . \ a_nb_n \ . \ . \ .$$

represents the product ab.

$$a_nb_n - ab = a_nb_n - a_nb + a_nb - ab.$$

Therefore,

$$|a_nb_n - ab| \overline{\overline{<}} |a_n||b_n - b| + |b||a_n - a|.$$

Now if M is some number larger than $|a|$, there is a place in the first sequence beyond which its elements are numerically less than M.

Again, for any positive number ε' there is a place in the first sequence beyond which its elements differ from a by less than ε' and a place in the second sequence beyond which its elements differ from b by less than ε'.

Beyond the farthest of all these places we have

$$|a_n| < M, \quad |a_n - a| < \varepsilon', \quad \text{and} \quad |b_n - b| < \varepsilon'.$$

Hence there is a place beyond which

$$|a_nb_n - ab| < M\varepsilon' + |b|\varepsilon',$$

and if ε' is taken so that

$$M\varepsilon' + |b|\varepsilon' < \varepsilon,$$

that is, if $$\varepsilon' < \frac{\varepsilon}{M+|b|},$$

we shall have a place in the product sequence beyond which its elements differ from ab by less than ε.

47. Theorem.—*If the two sequences*

$$a_1 \quad a_2 \;.\;.\;.\; a_n \;.\;.\;.$$

and $$b_1 \quad b_2 \;.\;.\;.\; b_n \;.\;.\;.\;.$$

represent the numbers a and b, respectively, and if the number a is not zero and none of the elements of the first sequence is zero, then the sequence

$$\frac{b_1}{a_1} \quad \frac{b_2}{a_2} \;.\;.\;.\; \frac{b_n}{a_n} \;.\;.\;.$$

represents the quotient $\dfrac{b}{a}$.

Proof. $$\frac{b_n}{a_n} - \frac{b}{a} = \frac{(ab_n - ab) - (a_nb - ab)}{aa_n}.$$

Therefore $$\left|\frac{b_n}{a_n} - \frac{b}{a}\right| \overset{=}{<} \frac{|a||b_n - b| + |b||a_n - a|}{|a||a_n|}.$$

a is not zero, and if M is a number lying between zero and $|a|$, there is a place in the sequence of a's beyond which all the elements are numerically greater than M. That is, there is a place beyond which

$$\frac{1}{|a_n|} < \frac{1}{M}.$$

Again, for any positive number ε' there is a place in the first sequence beyond which its elements differ from a by less than ε', and a place in the second sequence beyond which its elements differ from b by less than ε'.

Beyond the farthest of these three places

$$\left|\frac{b_n}{a_n} - \frac{b}{a}\right| < \frac{|a|\varepsilon' + |b|\varepsilon'}{|a|M}.$$

Now if ε' is taken so that the last expression shall be less than ε, that is, if

$$\varepsilon' < \frac{|a|M\varepsilon}{|a| + |b|},$$

we shall have a place beyond which

$$\left|\frac{b_n}{a_n} - \frac{b}{a}\right| < \varepsilon.$$

48. In other words, the operations of addition, subtraction, multiplication, and division of numbers represented by sequences, can be performed by performing these operations on corresponding elements of the sequences.

Any sequence may first be changed in accordance with the preceding theorems (Arts. 41 and 42), and thus the result may be presented in a great variety of forms. In particular, if we wish to divide by a number a which is not zero, and a is represented by a sequence some of whose elements are supposed to be zero, there can be only a finite number of these elements zero, and these may be omitted or changed into other numbers different from zero.

We can sometimes combine the corresponding elements of two sequences by addition, subtraction, multiplication, or division and produce a sequence which is regular, when the two given sequences are not regular.

Or we may divide the elements of a sequence by the corresponding elements of a sequence representing zero, if zero does not itself occur among the elements of the latter sequence, and when both sequences represent zero the resulting sequence may be regular and represent a number. But this number will have no special relation to the number zero, which the two sequences represent.

For example, dividing the elements of the sequence

$$1 \quad \frac{1}{3} \cdots \frac{1}{2n-1} \cdots$$

by the corresponding elements of the sequence

$$1 \quad \frac{1}{2} \cdots \frac{1}{n} \cdots,$$

both representing zero, we have a sequence

$$\cdot 1 \quad \frac{2}{3} \cdots \frac{n}{2n-1} \cdots,$$

which represents $\frac{1}{2}$.

NOTE.—The theory of irrational numbers given in Chapter I is due to Richard Dedekind; or, at least, most of its developments are due to Dedekind, and the writings of Dedekind have done the most to bring about its adoption at the present time. The reader will find the theory explained in an essay, "Continuity and Irrational Numbers," translated by W. W. Beman, and published by the Open Court Publishing Company. This theory has been adopted by Professor Fine in his "College Algebra."

There is another theory of irrational numbers in which the numbers are defined as regular sequences and the addition, multiplication, etc., of regular sequences serve as definitions of these operations on irrational numbers. This theory is due to Georg Cantor; some exposition of it may be found in Chapter IV of Professor Fine's "Number System of Algebra," and a detailed treatment in Professor Pierpont's "Theory of Functions of Real Variables."

The latter theory is more artificial than the theory of Dedekind. We naturally think of an irrational number as occupying a certain position among numbers, and in describing this position we mention other numbers between which it lies. Thus we say that the square root of 2 is greater than 1.414 and less than

1.415. We know the square root of 2 by being able to determine the position of any rational number with reference to it.

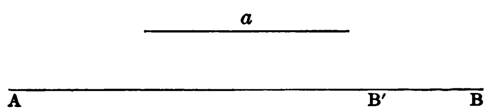
IV. The Theory of Limits

49. We naturally think of a variable as varying continuously, varying through a continuous set of values in a continuous interval of time; but in cases considered when we first study Algebra and Geometry there is no continuous set of values, nor is there any question of time. The values which the variable takes are distinct, each followed by one which comes next after it; in other words, they form a sequence, and when the variable in this way approaches a number a as limit its values form a sequence representing a. Thus it simplifies the subject to dismiss the idea of variable and fix our attention on the sequence.

However, as the language of variable and limit permeates nearly all of our mathematics, it may be well to consider a little in detail the meaning and use of these terms. What we shall say applies for the most part both when the variable is supposed to vary continuously and when it varies through a sequence of values.

50. A variable approaches a number a as limit if for every positive number ϵ there is a place among the values of the variable beyond which they all differ from a by less than ϵ.

51. The earliest application of the theory of limits that we meet is in connection with the ratio of incommensurable lines.



Let AB be a line incommensurable with a given line a. We divide a into some number of equal parts, say n equal parts. If AB is shorter than a, we suppose that n is taken large enough so that one of the n equal parts of a is shorter than AB.

One of these parts applied to AB will be contained in AB a certain number of times. Let it be contained m times, reaching to a point B' and leaving a remainder $B'B$ less than one of these parts. The line AB' is commensurable with a, and its ratio to a is the rational number $\frac{m}{n}$.

We repeat this process, each time dividing a into a larger number of parts. Each time the equal parts of a are smaller than before; and we can make them less than any given length by dividing a into a sufficiently large number of parts.

The remainder $B'B$ is always less than one of the equal parts of a, and, therefore, the remainder is a variable which approaches zero as limit.

The line AB' approaches AB as limit, and the ratio of AB' to a, the rational number $\frac{m}{n}$, is a variable which

approaches as limit the incommensurable ratio of AB to a.

In reality, we are simply forming a sequence of rational numbers in accordance with the theorem of Art. 33 to represent the irrational number which is the ratio of AB to a. If the incommensurable lines are the side and diagonal of a square, we have the very example of that article.

52. Certain things may be pointed out as not included in the above definition of limit.

(1) It is not necessary that the variable shall be always less or always greater than the limit.

Those who are familiar with continued fractions will recognize in the convergents a sequence of numbers alternately less and greater than the number represented. We may say that the nth convergent is a variable approaching a limit, its values alternately less and greater than the limit.

We shall have a similar illustration in Art. 71.

(2) It is not necessary that the variable shall be always approaching the limit, each value nearer than any which precedes.

In the case of the two incommensurable lines, we increase the number of parts into which a is divided, so that each part is made smaller. If the part is not made enough smaller to be contained one more time in AB, the line AB' will be smaller and the remainder $B'B$ larger than before. In fact, if we let the values of n be the successive integers, we may find that our variables move away from their limits more frequently than they approach them. This is shown in the ex-

ample of Art. 33. We can avoid this by taking n each time so large that one of the n equal parts of a shall be less than the last remainder, but this is not necessary in order that the line AB' shall approach AB and that the remainder shall approach zero as limit.

(3) It is not necessary that the variable shall never equal its limit.

This does not mean that in all cases a variable approaching a as limit will come finally to equal a, but that we are not to exclude by the definition certain cases where a is found among the values which the variable takes, or even certain cases where all of its values or all beyond a certain point are equal to a.

We can form a sequence representing a in which the number a occurs an infinite number of times among the elements of the sequence, by taking any sequence that represents a and interpolating elements equal to a. Such a sequence taken by itself would be in no way distinguishable from a sequence in which the number represented does not occur. A variable whose values are the elements of such a sequence would serve as a variable having a for limit just as well as if the number a were not found among its values.

53. We will change a little the illustration of the incommensurable lines.

Let us suppose that a line a diminishes continuously to the limit zero, varying through all values in order from some given length, each value which it takes being less than any preceding but not equal to zero.

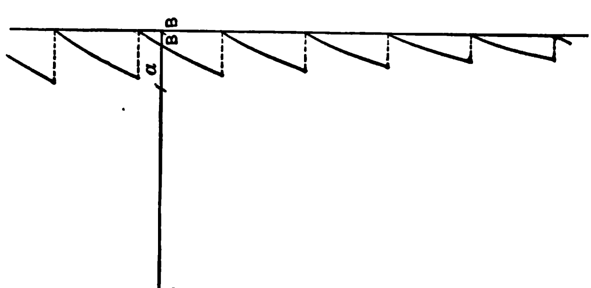
If we apply a to a line AB as many times as it will

go, we shall usually get a line AB' which is a multiple of a and a remainder $B'B$ less than a.

While the number of times that a is contained in AB remains the same, though a grows smaller, the line AB' will be continuously decreasing and the line $B'B$ will be continuously increasing. When a becomes just small enough to be contained one more time in AB the line AB' will become the same as AB and there will be no remainder. Then, as a continues to decrease, AB' will again diminish continuously, and there will be a remainder increasing continuously until a becomes small enough again to be contained in AB one more time without a remainder.

The line AB' and the remainder $B'B$ are in this case variables which take the values AB and zero an infinite number of times, and which at all other times are constantly moving away from these values. Yet they are variables approaching these values as limits; for, given any length, there is a point in the process beyond which they always differ from these limits by less than the given length.

The following diagram represents the way in which these variables approach their limits.



54. There are certain things which are essential to an understanding of the Theory of Limits, and which should be emphasized.

(1) It must be possible not only to make the difference between the variable and its limit less than any assigned positive number, but also to make it remain less than the assigned number.

(2) The "assigned positive number" * means every positive number, and the place beyond which the variable differs from its limit by less than the assigned number is not necessarily the same for two different numbers. Indeed, no one value of the variable differs from its limit by less than any number whatever unless it is equal to the limit itself.

(3) We must distinguish between the question whether a variable approaches some given number as limit and the question whether the variable approaches a limit. In one case we have to consider the differences between the given number and the values of the variable; in the other case we must first determine

---

* Sometimes the words "however small" are added. It is not correct, however, to say of any number that it is a small number or that it is a large number. We can only say of two different numbers that one is smaller than the other and that one is larger than the other. We must not think that our system contains, besides ordinary numbers, another class of numbers that are indefinitely small, and that the difference between a variable and its limit finally gets out of the region of ordinary numbers and into the region of these numbers. A variable approaching zero as limit is sometimes called an infinitesimal, but its values are not infinitesimal numbers; they are all ordinary numbers, like 2 or $\sqrt{3}$ or $\frac{1}{7}$. We have used ε to denote every positive number, and when we might say "every positive number however large" (e.g., in Art. 19) we mean the same thing and so we use the same symbol.

a number which we can then prove is the limit, or we must prove that there is no such number.

Thus we prove, in particular, the two following theorems (see Arts. 37 and 40):

Theorem.—*If a variable is constantly increasing and is always less than some fixed number M, the variable approaches a limit, either M or some smaller number.*

Theorem.—*A variable approaches a limit if for every ε there is a place among its values beyond which they all differ from one another by less than ε.*

55. In some of our text-books much stress is laid on what is called The Theorem of Limits: If two variables are constantly equal and approach limits their limits are equal. This only means that the same variable cannot approach as limit two different numbers—that the variable determines the limit. In fact, the variable could never differ from both of two different numbers by less than a half of their difference (see Art. 28).

In proving what is called the "incommensurable case" of certain propositions of geometry we have two incommensurable ratios which we wish to prove equal. We may do this by constructing for each a sequence of commensurable ratios as explained in Art. 51, every ratio in one sequence being the same rational number as the corresponding ratio in the other; that is, we construct a single sequence of rational numbers which represents both incommensurable ratios. Therefore, as a sequence cannot represent two different numbers,

the two incommensurable ratios must be the same irrational number. In the language of limits we say that a single variable has both of these ratios for limit, and that these ratios are therefore equal, since a variable cannot have two different limits. However, if we define the ratio of two incommensurable magnitudes as the irrational number which occupies the point of separation of all rational numbers into two classes, we can prove the proposition directly, making no use of sequences or of the theory of limits (see Art. 8).

CHAPTER III

SERIES

I. Convergency of Series

56. A SERIES is a sequence written in a form which presents the differences of successive elements as a succession of terms.

Suppose we have a sequence

$$a_1 \quad a_2 \;.\;.\;.\; a_n \;.\;.\;.\;,$$

and put

$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_2 - a_1 \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ u_n &= a_n - a_{n-1} \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;; \end{aligned}$$

then

$$\begin{aligned} a_1 &= u_1 \\ a_2 &= u_1 + u_2 \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_n &= u_1 + u_2 + \;.\;.\;.\; + u_n \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{aligned}$$

and we express the sequence by writing

$$u_1 + u_2 + \;.\;.\;.\; + u_n + \;.\;.\;.\;.$$

This is called a series. The numbers $u_1 \ u_2 \ldots u_n \ldots$ are called its terms.

The sequence

$$a_1 \quad a_2 \ . \ . \ . \ a_n \ . \ . \ .$$

written in the form of a series becomes

$$a_1+(a_2-a_1)+ \ . \ . \ . \ +(a_n-a_{n-1})+ \ . \ . \ . \ .$$

The series

$$u_1+u_2+ \ . \ . \ . \ +u_n+ \ . \ . \ .$$

written in the form of a sequence becomes

$$u_1, \quad u_1+u_2, \ . \ . \ . \ , \quad u_1+u_2+ \ . \ . \ . \ +u_n, \ . \ . \ . \ .$$

Since the nth element of the sequence is the sum of the first n terms of the series, it will often be convenient to write this s_n. Thus we shall say that the series

$$u_1+u_2+ \ . \ . \ . \ +u_n+ \ . \ . \ .$$

is the same as the sequence

$$s_1 \quad s_2 \ . \ . \ . \ s_n \ . \ . \ . \ ,$$

where

$$s_n=u_1+u_2+ \ . \ . \ . \ +u_n.$$

A series is convergent when the corresponding sequence represents a number, and this number is called the value of the series.

For example, the sequence

$$\frac{1}{2} \quad \frac{2}{3} \quad \cdots \quad \frac{n}{n+1} \quad \cdots$$

is the same as the series

$$\frac{1}{1.2}+\frac{1}{2.3}+ \quad \cdots \quad +\frac{1}{n(n+1)}+ \quad \cdots.$$

The sequence represents 1 (p. 37); therefore the series is convergent and its value is 1.

A series which is not convergent is divergent.

57. Theorem.—*The geometrical series*

$$a+ar+ \quad \cdots \quad +ar^{n-1}+ \quad \cdots$$

is convergent when r is numerically less than 1, and its value is $\frac{a}{1-r}$.

Proof.—The sum of the first n terms is

$$\frac{a-ar^n}{1-r}.$$

That is, the series is the same as the sequence

$$a \quad \frac{a-ar^2}{1-r} \quad \cdots \quad \frac{a-ar^n}{1-r} \quad \cdots.$$

The difference between the nth element of this sequence

and the number $\frac{a}{1-r}$ is the numerical value of the fraction

$$\frac{ar^n}{1-r}.$$

There is a place among the powers of r beyond which they are all numerically less than ϵ', ϵ' being any given positive number. By taking ϵ' so that

$$\left|\frac{a}{1-r}\right|\epsilon' < \epsilon,$$

that is, so that

$$\epsilon' < \epsilon\left|\frac{1-r}{a}\right|,$$

we have a place in the sequence beyond which its elements all differ from $\frac{a}{1-r}$ by less than ϵ.

58. Theorem.—*If a series is convergent, its terms themselves form a sequence equal to zero.*

Proof.—In a sequence which represents a number and so is regular, there is a place for each ϵ beyond which all the elements differ from each other by less than ϵ. Now the terms of the series are the differences of successive elements of the corresponding sequence. Therefore if the series is convergent, there is for each ϵ a place in the series beyond which all the terms are numerically less than ϵ.

The converse is not necessarily true. For an example see Art. 61.

59. Theorem.—*If a series is convergent, then for every ε there is a place in the series beyond which any sum of successive terms is less than ε.*

Proof.—Any sum of successive terms is the difference between some two elements of the sequence; and if the sequence represents a number and so is regular, there is a place beyond which any two elements differ by less than ε.

Thus if the series is

$$u_1+u_2+\ .\ .\ .\ +u_n+\ .\ .\ .\ ,$$

and the corresponding sequence

$$s_1\quad s_2\ .\ .\ .\ s_n\ .\ .\ .\ ,$$

the sum of p successive terms beginning with the $(n+1)$th will be

$$u_{n+1}+u_{n+2}+\ .\ .\ .\ +u_{n+p}=s_{n+p}-s_n.$$

60. Theorem.—*Conversely, a series is convergent if for every ε there is a place in the series beyond which every sum of successive terms is less than ε.*

Proof.—Every difference of elements of the corresponding sequence is a sum of successive terms of the series; and if for every ε there is a place beyond which

these differences are all less than ϵ, the sequence is regular and the series convergent.

61. Theorem.—*The series*

$$1+\frac{1}{2}+\ .\ .\ .\ +\frac{1}{n}+\ .\ .\ .$$

is not convergent.

Proof.—The sum of p successive terms beginning with the $(n+1)$th is

$$\frac{1}{n+1}+\frac{1}{n+2}+\ .\ .\ .\ +\frac{1}{n+p}.$$

The last of these being the smallest, their sum is greater than $\frac{p}{n+p}$. Therefore for a value of ϵ less than 1 there is no place in the series beyond which every sum of successive terms is less than ϵ. For example, whatever the value of n, if we take p greater than n we shall have $\frac{p}{n+p}$ greater than $\frac{1}{2}$.

If we take two or more terms of this series, and then take as many more, we shall increase what we had obtained before by more than $\frac{1}{2}$. If we take as many more as we have now, we shall again increase what we have by more than $\frac{1}{2}$. Since there is no limit to the number of times this may be done, there is no

limit to the number which we may obtain by taking a sufficient number of terms of this series.

This series is called the harmonic series.

62. **Theorem.**—*A series is convergent if the series formed by taking the numerical values of its terms is convergent.*

Proof.—The sum of any number of successive terms is numerically equal to or less than the sum of the numerical values of these terms. Since the series of numerical values is convergent, there is a place in it beyond which any sum of successive terms is less than ε. The same must be true of the given series, and the given series is convergent.

This theorem may also be stated as follows:

Theorem.—*If a series of positive terms is convergent, it will remain so after the signs of any portion of its terms have been changed.*

Another proof will be given on page 89.

A series which remains convergent when we replace its terms by their numerical values is called an absolutely convergent series.

A convergent series which becomes divergent when its terms are replaced by their numerical values is called a semi-convergent series.

Cor.—An absolutely convergent series is numerically equal to or less than the series of numerical values of its terms.

For the sum of its first n terms is numerically equal

to or less than the sum of their numerical values (see Art. 36).

63. Theorem.—*It does not affect the convergence of a series to change any finite number of its terms.*

Proof.—If the nth term of the series is the last term changed, all the elements of the sequence beyond the nth, containing as they do the first n terms, are increased or diminished by the same quantity, and their differences, which alone determine whether the sequence is regular, are not changed.

Thus many of our theorems will be true when the conditions do not hold throughout the series, but only beyond a certain place.

64. Theorem.—*A series of positive terms is convergent if the elements of the corresponding sequence are all less than some fixed number M.*

Proof.—In a series of positive terms each element of the corresponding sequence is greater than the preceding. If, then, each element is also less than M, the sequence represents a number equal to M or less than M (Art. 37).

In other words, if the sum of the first n terms of a series of positive terms is less than some number M, the same for all values of n, the series is convergent.

Cor.—If a series of positive terms is not convergent, then for every ϵ there is a place in the corresponding sequence beyond which all the elements are greater than ϵ.

Such a series may especially be called a divergent series, but we usually call any series divergent which is not convergent.

63. **Theorem.**—*The series*

$$1+\frac{1}{2^p}+\ \ldots\ +\frac{1}{n^p}+\ \ldots$$

is convergent when p is any number greater than 1.

Proof.—Let s_n denote the sum of the first n terms of this series; that is, let the corresponding sequence be

$$s_1\quad s_2\ \ldots\ s_n\ \ldots\,.$$

The sum of m terms beginning with the mth is

$$s_{2m-1}-s_{m-1}=\frac{1}{m^p}+\ \ldots\ +\frac{1}{(2m-1)^p}.$$

The first of these terms is the largest. Therefore their sum is less than $\frac{m}{m^p}$ or $\frac{1}{m^{p-1}}$.

Putting $m=2,\ 4,\ 8,\ \ldots,$ we have

$$s_3<s_1+\frac{1}{2^{p-1}}=1+\frac{1}{2^{p-1}},$$

$$s_7<s_3+\frac{1}{4^{p-1}}<1+\frac{1}{2^{p-1}}+\left(\frac{1}{2^{p-1}}\right)^2,$$

$$s_{15}<1+\frac{1}{2^{p-1}}+\left(\frac{1}{2^{p-1}}\right)^2+\left(\frac{1}{2^{p-1}}\right)^3,$$

etc.

If $n<2^r$, s_n is less than the sum of the first r terms of the geometrical series

$$1+\frac{1}{2^{p-1}}+\ .\ .\ .\ +\left(\frac{1}{2^{p-1}}\right)^{r-1}+\ .\ .\ .\ ,$$

which is convergent, p being greater than 1. Whatever the value of n, we can take r so that $2^r>n$ (Art. 19); and, therefore, for all values of n, s_n is less than the value of this geometrical series, namely,

$$s_n<\frac{1}{1-\frac{1}{2^{p-1}}}.$$

This is a fixed number, and the given series, being a series of positive terms, is convergent.

66. Theorem.—*If a series of positive terms is convergent, the sum of any terms selected from it is less than the value of the series.*

Proof.—Let u_n be the last of those selected, and s_n the sum of the first n terms. The terms selected are all included in s_n and their sum is equal to or less than s_n. But s_n is less than the number represented by the series (see proof on page 45). Therefore the sum of the terms selected is less than the value of the series.

67. Theorem.—*If a series of positive terms is convergent, any series formed by taking a part of its terms is convergent.*

Proof.—The first n terms of such a series are found somewhere in the original series, and their sum is less than the value of the original series. Therefore, as the sum of the first n terms is less than a certain fixed number, the series is convergent.

By virtue of Art. 62 the theorem may be stated as follows:

Theorem.—*If a series is absolutely convergent, any series formed by taking a part of its terms is absolutely convergent.*

Cor.—If a series of positive terms is convergent, or if a series is absolutely convergent, there is a place beyond which the sum of any terms selected from it, or the value of any series formed by taking a part of its terms, is numerically less than ϵ.

Proof.—The series formed by omitting the first n terms of a convergent series is equal to the difference between the nth element of the corresponding sequence and the number which the sequence represents, and there is a place beyond which this difference is less than ϵ. When a series is absolutely convergent this is true of the series of numerical values of its terms, and there is a place beyond which the remainder series, the sum of any terms selected from it, or any series formed by taking a part of its terms, will have a value numerically less than ϵ.

68. Theorem.—*If the terms of a series of positive terms are equal to or less than the corresponding terms of another*

series of positive terms which is convergent, then the given series is convergent.

Proof.—The sum of the first n terms of the given series is equal to or less than the sum of the first n terms of the series which is convergent, and therefore less than the value of the latter series. That is, we have a series of positive terms, and the sum of the first n terms is less than a certain fixed number. Hence the series is convergent.

Cor.—If the terms of a series of positive terms are equal to or greater than the corresponding terms of another series of positive terms which is divergent, the given series is divergent.

For if the given series were convergent, the other series would be convergent by the theorem.

The theorem and corollary are true whenever there is a place in the series beyond which the conditions are satisfied (Art. 63).

In most cases we determine the convergence of a series by comparing it with other series. Two of the simplest series for purposes of comparison are the series

$$1+\frac{1}{2}+\ \ldots\ +\frac{1}{n}+\ \ldots,$$

which is divergent, and the series

$$1+\frac{1}{2^2}+\ \ldots\ +\frac{1}{n^2}+\ \ldots,$$

which is convergent.

By virtue of Art. 62 the theorem may also be stated as follows:

Theorem.—*If in any series there is a place beyond which the terms are numerically equal to or less than the corresponding terms of another series which is absolutely convergent, the given series is absolutely convergent.*

69. Theorem.—*The series*

$$1+\frac{1}{2^p}+\ .\ .\ .\ +\frac{1}{n^p}+\ .\ .\ .$$

is divergent for any value of p equal to or less than 1.

Proof.—When $p=1$ the series is the harmonic series already proved divergent, and when $p<1$, whether positive or negative, all the terms of the series are greater than the corresponding terms of the harmonic series, except the first, which is the same in the two series.

70. Theorem.—*A series of positive terms is convergent if the ratio of each term to the preceding is less than some fixed positive number which is itself less than 1.*

Proof.—Let the series be

$$u_1+u_2+\ .\ .\ .\ +u_n+\ .\ .\ .\ ,$$

with all of its terms positive, and suppose we have

$$\frac{u_2}{u_1}<r,\ \ \frac{u_3}{u_2}<r,\ .\ .\ .\ ,\ \ \frac{u_n}{u_{n-1}}<r,\ .\ .\ .\ ,$$

where r is a fixed number less than 1.

Multiplying together the corresponding members of the first $n-1$ of these inequalities (Art. 16, Cor.), we have

$$\frac{u_2 \ . \ . \ . \ u_n}{u_1 \ . \ . \ . \ u_{n-1}} < r^{n-1},$$

or
$$\frac{u_n}{u_1} < r^{n-1};$$

whence, since u_1 is positive,

$$u_n < u_1 r^{n-1}.$$

The terms of our series are, therefore, less than the corresponding terms of the geometrical series

$$u_1 + u_1 r + \ . \ . \ . \ + u_1 r^{n-1} + \ . \ . \ . \ ,$$

except that the first terms are the same in the two series. The latter series is convergent, since $r<1$; therefore the given series is convergent.

By virtue of Arts. 62 and 63 the theorem may be stated as follows:

Theorem.—*If in any series there is a place beyond which the ratio of each term to the preceding is numerically less than some fixed number which is itself less than 1, the series is absolutely convergent.*

Cor.—If in the series

$$u_1 + u_2 + \ . \ . \ . \ + u_n + \ . \ . \ .$$

the sequence of ratios

$$\frac{u_2}{u_1} \quad \frac{u_3}{u_2} \quad . \; . \; . \quad \frac{u_n}{u_{n-1}} \quad . \; . \; .$$

represents a number numerically less than 1, the series is absolutely convergent.

Proof.—Let l be the numerical value of the number which this sequence represents, and let r be a number between l and 1. Since $r>l$ there is a place in this sequence beyond which its elements are numerically less than r (Art. 35). Therefore, by the preceding statement of the theorem, the series is absolutely convergent.

71. Theorem.—*If the terms of a series are alternately positive and negative and each is numerically less than the preceding, and if they form a sequence equal to zero, the series is convergent.*

Proof.—Let the series be

$$u_1 - u_2 + u_3 - \; . \; . \; . \; + u_{2n-1} - u_{2n} + \; . \; . \; . \; ,$$

and the corresponding sequence

$$s_1 \quad s_2 \quad s_3 \; . \; . \; . \; s_{2n-1} \quad s_{2n} \; . \; . \; . \; .$$

The elements of the sequence are alternately less and greater each than the preceding, but when we subtract a number u we then add a smaller number, and

when we add a number u we then subtract a smaller number. Thus each element of the sequence lies between the preceding two, and between any two successive elements of the sequence lie all that follow.

Now the difference between any two successive elements of the sequence is a term of the series, and from the last part of our hypothesis it follows that there are terms of the series numerically less than ϵ. Hence there is a place in the sequence beyond which the elements differ from each other by less than ϵ, and the sequence is regular.

Another proof is given on page 83.

The odd-numbered elements of the sequence form by themselves a sequence of elements each less than the preceding and greater than any one of the even-numbered elements of the sequence, and the even-numbered elements of the sequence form by themselves a sequence of elements each greater than the preceding and less than any one of the odd-numbered elements. This would be true, and the two sequences would be convergent, even if the terms of the series did not form a sequence equal to zero, but the two sequences would then represent different numbers.

Cor.—The value of the series of the theorem lies between any two successive elements of the sequence; thus the sum of any number of terms from the beginning differs from the value of the series by less than the next term.

An example is the series

$$1-\frac{1}{2}+\frac{1}{3}-\ \ldots\ +\frac{1}{2n-1}-\frac{1}{2n}+\ \ldots\ .$$

By virtue of Art. 63 the theorem and corollary may be stated as follows:

Theorem.—*If there is a place in a series beyond which the terms are alternately positive and negative, each term less than the preceding, and if the terms form a sequence equal to zero, the series is convergent.*

Cor.—The value of the series lies between any two successive elements of the sequence beyond the place where the conditions of the theorem become true.

II. Operations upon Series

72. Theorem.—*If a series is convergent, its terms may be grouped in parentheses in any manner without destroying its convergence or changing its value.*

Proof.—This is the same as omitting elements from the corresponding sequence (Art. 41).

73. Theorem.—*When the terms of a convergent series are grouped in parentheses it does not change its value to remove the parentheses.*

Proof.—This is but another way of stating the preceding theorem.

Removing the parentheses is the same as interpolating elements in the corresponding sequence.

In particular, if a series of parentheses is convergent we may remove the parentheses

(1) When all the terms in each parenthesis have the same sign, or

(2) When the number of terms in each parenthesis is less than some fixed number and the terms of the new series form a sequence equal to zero.

Proof.—Let m be the number of terms of the series of parentheses included entirely in the first n terms of the new series. That is, the first n terms of the new series include all of the terms of the first m parentheses and perhaps some of the terms from the next parenthesis.

The sum of the first n terms of the new series will differ from the sum of the first m parentheses by zero or by a sum of terms from the $(m+1)$th parenthesis.

Now, in case (1), there is a place in the series of parentheses beyond which they are numerically less than $\frac{\varepsilon}{2}$, and any sum of terms from a single parenthesis, being equal to or less than the entire parenthesis, will be less than $\frac{\varepsilon}{2}$.

Again, in case (2), where the number of terms in each parenthesis is less than a fixed number, say p, since the terms form a sequence equal to zero, there is a place beyond which they are numerically less than $\frac{\varepsilon}{2p}$, and beyond which, therefore, any sum of terms

from a single parenthesis will be numerically less than $\frac{\epsilon}{2}$.

Thus, in either case, there is a place beyond which we can say that the sum of the first n terms of the new series differs from the sum of the first m terms of the series of parentheses by less than $\frac{\epsilon}{2}$. But as the series of parentheses is convergent there is a place beyond which we can say that the sum of its first m terms differs from the value of the series by less than $\frac{\epsilon}{2}$. Beyond the farther of these two places we can say that the sum of the first n terms of the new series differs from the value of the series of parentheses by less than ϵ.

As an illustration we may give the following proof of the theorem of Art. 71:

The series

$$(u_1-u_2)+\ \ldots\ +(u_{2n-1}-u_{2n})+\ \ldots$$

is a series of positive terms. The sum of its first n terms may be written

$$u_1-(u_2-u_3)-\ldots-(u_{2n-2}-u_{2n-1})-u_{2n},$$

where the expression in any parenthesis represents a positive number. Hence this sum is less than the fixed number u_1 and the series of parentheses is convergent.

Now, if the u's form a sequence equal to zero, the parentheses may be removed by the second case above. Therefore the given series is convergent.

74. Theorem.—*The series formed by adding the corresponding terms of two convergent series is convergent and equal to the sum of their values; and the series formed by subtracting the terms of one convergent series from the corresponding terms of a second convergent series is convergent, and its value is the difference obtained by subtracting the value of the first series from the value of the second.*

Proof.—This is the same as adding or subtracting the corresponding elements of two regular sequences; the resulting sequence represents the sum or difference of the numbers represented by the two sequences.

We may group the terms of two convergent series in parentheses in any manner and after adding or subtracting remove the parentheses, and the resulting series will still be convergent and represent the sum or difference of the two given series.

For the sum of any number of terms of the final series will differ from the sum of a certain number of terms before the parentheses are removed by zero or by a sum of terms from a single parenthesis. But the sum of terms from a single parenthesis here is in any case the sum of a certain number of successive terms from one or the other of the original series, and there is a place in each of the original series beyond which any such sum of terms is numerically less than ε. Therefore we may remove the parentheses, as in the second case considered in the last article.

Cor.—If we combine by addition or subtraction a convergent and a divergent series, the resulting series will be divergent.

For that one of the two series which is divergent could be obtained by combining the resulting series with the other given series, and if the resulting series were convergent, we should have a divergent series as the difference or sum of two convergent series, which is contrary to the theorem.

75. Theorem.—*If a series is convergent and equal to a number S, the series formed by multiplying all of its terms by a number m will be convergent and equal to mS.*

Proof.—This is the same as multiplying together two sequences, one representing S and the other composed of elements all equal to m.

Cor.—A convergent series equal to S becomes a convergent series equal to $-S$ if we change the signs of all of its terms.

76. Theorem.—*If a series of positive terms is convergent, the series will be convergent which we form when we multiply its terms by the corresponding elements of a sequence of positive numbers all less than some positive number m.*

Proof.—The sum of n terms of the new series will be less than the sum of n terms of a series formed by multiplying the terms of the original series by m, and so less than mS, where S is the value of the original series. Therefore the new series will be convergent.

By virtue of Art. 62 we may state the theorem as follows:

Theorem.—*If a series is absolutely convergent, the series will be absolutely convergent which we get when we multiply its terms by the corresponding elements of a sequence of numbers all numerically less than some fixed number m.*

Cor.—An absolutely convergent series will remain so if we multiply its terms by the corresponding elements of a regular sequence.

77. Theorem.—*If two series are absolutely convergent, a series can be formed by multiplying every term of one with every term of the other that will be absolutely convergent and will equal the product of the values of the given series.*

Proof.—Let the two series be

$$u_1 + u_2 + \dots + u_n + \dots$$

and
$$v_1 + v_2 + \dots + v_n + \dots,$$

and let the corresponding sequences be

$$a_1 \quad a_2 \dots a_n \dots$$

and
$$b_1 \quad b_2 \dots b_n \dots .$$

The product of these sequences is the sequence

$$a_1 b_1 \quad a_2 b_2 \dots a_n b_n \dots ,$$

and this is equivalent to the series whose successive terms are

$$a_1b_1 \quad a_2b_2 - a_1b_1 \ . \ . \ . \ a_nb_n - a_{n-1}b_{n-1} \ . \ . \ . \ ,$$

that is, to the series

$$(\alpha) \qquad \begin{aligned} &u_1v_1 + (u_1v_2 + u_2v_1 + u_2v_2) + \ . \ . \ . \\ &\quad + (u_1v_n + u_2v_n + \ . \ . \ . \ + u_{n-1}v_n \\ &\qquad\qquad + u_nv_1 + u_nv_2 + \ . \ . \ . \ + u_nv_{n-1} \\ &\qquad\qquad\qquad + u_nv_n) + \ . \ . \ . \ . \end{aligned}$$

Now the series of numerical values

$$|u_1| + |u_2| + \ . \ . \ . \ + |u_n| + \ . \ . \ .$$

and $$|v_1| + |v_2| + \ . \ . \ . \ + |v_n| + \ . \ . \ .$$

are equivalent to two sequences which we will write

$$a_1' \quad a_2' \ . \ . \ . \ a_n' \ . \ . \ .$$

and $$b_1' \quad b_2' \ . \ . \ . \ b_n' \ . \ . \ . \ .$$

The product is the sequence

$$a_1'b_1' \quad a_2'b_2' \ . \ . \ . \ a_n'b_n' \ . \ . \ . \ ,$$

equivalent to the series

$$(\alpha') \qquad \begin{aligned} &|u_1v_1| + (|u_1v_2| + |u_2v_1| + |u_2v_2|) + \ . \ . \ . \\ &\quad + (|u_1v_n| + |u_2v_n| + \ . \ . \ . \ + |u_{n-1}v_n| \\ &\qquad\qquad + |u_nv_1| + |u_nv_2| + \ . \ . \ . \ + |u_nv_{n-1}| \\ &\qquad\qquad\qquad + |u_nv_n|) + \ . \ . \ . \ . \end{aligned}$$

In the series (α') all the terms in any parenthesis have the same sign. Therefore the parentheses may be removed without destroying its convergence or changing its value.

The following series is, therefore, convergent:

$$
(\beta') \qquad \begin{aligned} &|u_1v_1|+|u_1v_2|+|u_2v_1|+|u_2v_2|+\ \ldots \\ &\qquad +|u_1v_n|+|u_2v_n|+\ \ldots \\ &\qquad\qquad +|u_nv_1|+\ \ldots\ +|u_nv_n|+\ \ldots . \end{aligned}
$$

The terms of (β') are the numerical values of the terms of the series

$$
(\beta) \qquad \begin{aligned} &u_1v_1+u_1v_2+u_2v_1+u_2v_2+\ \ldots \\ &\qquad +u_1v_n+u_2v_n+\ \ldots \\ &\qquad\qquad +u_nv_1+\ \ldots\ +u_nv_n+\ \ldots . \end{aligned}
$$

The series (β) is, then, absolutely convergent. Its value is not changed if we group the terms in any manner, for example, in the manner of the series (α). The value of the series (β) is, therefore, the same as that of the series (α); it is the product of the values of the two given series.

The series (α) represents the product of two given series when they are convergent, even if they are not both absolutely convergent.

III. Absolute Convergence

78. We have proved that a convergent series of positive terms will remain convergent after the signs of any portion of its terms have been changed (Art. 62). We may also prove this theorem as follows:

The terms which are not changed form by themselves a convergent series, and the terms which are changed form by themselves a convergent series (Art. 67). The original series is the sum of these two series, and the new series is formed by subtracting the second from the first.

79. Theorem.—*In a semi-convergent series the positive terms taken by themselves form a divergent series and the negative terms taken by themselves form a divergent series.*

Proof.—If the two part-series were both convergent, the series formed by taking all the terms with positive sign would be convergent and would be equal to the sum of the numerical values of the two series.

If one of the part-series were convergent and the other divergent, the given series, formed by taking them with positive and negative signs, respectively, would be divergent (Art. 74, Cor.), which is contrary to the hypothesis that the given series is convergent

Therefore both part-series must be divergent.

80. Theorem.—*If in each of two divergent series of positive terms the terms themselves form a sequence equal to zero, the two series may be put together, one with positive signs and the other with negative signs, in such a manner*

as to form a semi-convergent series equal to any given number M.

Proof.—We may, for example, take just enough positive terms from one series to make more than M, then just enough negative terms from the other series to make less than M, then just enough positive terms from the first series again to make more than M, and so on (Art. 64, Cor.). The result at each stage will differ from M by an amount equal to or less than the larger of the last terms taken from the two series. Therefore the series so formed will equal M.

Although it might be quite impossible to write down a formula for the nth term of the resulting series, the term in each place is determined by the above process.

When two series contain the same terms in different orders, every term of one being found somewhere in the other, one series is said to be obtained from the other by changing the order of its terms. This does not mean a change of order like that referred to on page 36. In any change in the order of the terms, the terms in the new arrangement must still form a sequence, each term having a definite numbered position with only a finite number of terms coming before it.

In the proof of the theorem the order of the terms in each of the two given series is supposed to be unchanged. But a divergent series of positive terms can never become convergent by a change in the order of its terms (see next theorem), and therefore it is not

necessary to suppose the order of the terms in the two given series to be unchanged.

Cor.—A semi-convergent series may be made to take any value M, by properly changing the order of its terms.

For examples see Art. 83.

81. Theorem.—*We may change the order of terms in an absotulely convergent series in any manner without destroying its convergence or changing its value.*

Proof.—First let all the terms of the series be positive, and let S be its value. If we form a new series by changing the order of its terms, the first n terms of the new series will be found somewhere in the original series, and their sum will be less than S (Art. 66). The new series is, then, convergent, and its value, say S', is equal to or less than S.

But we may obtain the original series from the new series by changing the order of its terms. Thus S must also be equal to or less than S'. Both of these conditions can only be true if S and S' are equal. Therefore any change in the order of terms of a convergent series of positive terms does not change its convergence or its value.

Now an absolutely convergent series of positive and negative terms is equal to the difference of two convergent series of positive terms. Any change in the order of its terms may produce a change in the order cf the terms of one or both of these two series, but such a change does not affect their convergence

or their values. Therefore the new series is convergent and its value is the same as that of the original series.

The expression "conditional convergence" is sometimes used with reference to the relation which the order of the terms of a series has to its convergence and value. A series which is convergent and has the same value whatever the order of its terms is unconditionally convergent. A series which sometimes loses its convergence, or at least changes its value when the order of its terms is changed, is conditionally convergent.

82. The product of two absolutely convergent series as given in Art. 77 is an absolutely convergent series, and its terms may be written in any order. One arrangement which is often useful is obtained by putting in succession those terms for which the sum of the subscripts of u and v is the same number.

That is, the two given series being

$$u_1+u_2+\ .\ .\ .\ +u_n+\ .\ .\ .$$

and

$$v_1+v_2+\ .\ .\ .\ +v_n+\ .\ .\ .\ ,$$

both absolutely convergent, their product may be written

$$u_1v_1+u_1v_2+u_2v_1+u_1v_3+u_2v_2+u_3v_1+\ .\ .\ .\ .$$

Any number of absolutely convergent series may be multiplied together in the same way. The product forms an absolutely convergent series whose terms may be arranged in any order.

83. Examples of changes in the order of terms of semi-convergent series.

First we will consider some cases where there is no change in the order of the positive terms among themselves or of the negative terms among themselves, but only in the order in which they are put together.

Take the series

$$1-\frac{1}{\sqrt{2}}+\ .\ .\ .\ +\frac{1}{\sqrt{2n-1}}-\frac{1}{\sqrt{2n}}+\ .\ .\ .\ .$$

(Arts. 69 and 71).

Let s_n denote the sum of the first n terms and S the value of this series, and let s_n' denote the sum of the first n terms when the series is arranged in some other order.

Let n' denote the number of positive terms and n the number of negative terms in the first $n'+n$ terms of the new arrangement. The values of n' and n are determined for each value of $n'+n$.

In $s'_{n'+n}$ the positive terms are

$$1+\frac{1}{\sqrt{3}}+\ .\ .\ .\ +\frac{1}{\sqrt{2n'-1}}$$

and the negative terms

$$-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{4}}-\ .\ .\ .\ -\frac{1}{\sqrt{2n}}.$$

If $n'>n$, $s_{2n'}$ will contain all of these terms and in

addition the negative terms

$$-\frac{1}{\sqrt{2(n+1)}}- \quad . \quad . \quad . \quad -\frac{1}{\sqrt{2n'}}.$$

Hence

$$s'_{n'+n}-s_{2n'}=\frac{1}{\sqrt{2(n+1)}}+ \quad . \quad . \quad . \quad +\frac{1}{\sqrt{2n'}}.$$

These terms are $n'-n$ in number. They are all less than $\frac{1}{\sqrt{2n}}$ and their sum is less than $\frac{n'-n}{\sqrt{2n}}$. The smallest of these terms is the last, and their sum is greater than $\frac{n'-n}{\sqrt{2n'}}$. (Serret)

Suppose, for example,

$$n'-n \overline{\overline{>}} \sqrt{an} \quad \text{and} \quad <\sqrt{an}+1,$$

a being some positive number. Then

$$\frac{n'-n}{\sqrt{2n}}<\frac{\sqrt{an}+1}{\sqrt{2n}}=\sqrt{\frac{a}{2}}+\frac{1}{\sqrt{2n}},$$

and

$$\frac{n'-n}{\sqrt{2n'}}\overline{\overline{>}}\frac{\sqrt{an}}{\sqrt{2n'}}=\sqrt{\frac{a}{2}}\cdot\sqrt{\frac{n}{n'}},$$

or, as $n'<n+1+\sqrt{an}$,

$$\frac{n'-n}{\sqrt{2n'}}>\sqrt{\frac{a}{2}}\cdot\sqrt{\frac{n}{n+1+\sqrt{an}}}=\sqrt{\frac{a}{2}}\frac{1}{\sqrt{1+\frac{1}{n}+\sqrt{\frac{a}{n}}}}.$$

That is,

$$s'_{n'+n}-s_{2n'}<\sqrt{\frac{a}{2}}+\frac{1}{\sqrt{2n}},$$

and

$$>\sqrt{\frac{a}{2}}\frac{1}{\sqrt{1+\frac{1}{n}+\sqrt{\frac{a}{n}}}}.$$

Both of these expressions, for integer values of n, form sequences representing $\sqrt{\frac{a}{2}}$. Therefore the sequence of s''s is regular and its value is

$$S'=S+\sqrt{\frac{a}{2}}.$$

If $a=2$, corresponding values of n and n' are

$$\begin{array}{llllllllll} n=1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\ n'=3 & 4 & 6 & 7 & 9 & 10 & 11 & 12 & 14 & \ldots. \end{array}$$

$$S'=1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{9}}$$
$$+\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{6}}+\ldots$$
$$=S+1.$$

Again, take the series

$$1-\frac{1}{2}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}+\ldots.$$

Using the same notation, if we suppose, for example,

$$n'-n \overline{\overline{>}} an \quad \text{and} \quad < an+1,$$

where a is some positive number, we shall find that

$$S' \overline{\overline{<}} S+\frac{a}{2} \quad \text{and} \quad \overline{\overline{>}} S+\frac{a}{2(1+a)}.$$

With this particular series we may proceed also as follows:

Write

$$\alpha_n = 1+\frac{1}{2}+ \ . \ . \ . \ +\frac{1}{n}.$$

Then
$$\frac{\alpha_n}{2}=\frac{1}{2}+\frac{1}{4}+ \ . \ . \ . \ +\frac{1}{2n},$$

$$\alpha_{2n}-\frac{\alpha_n}{2}=1+\frac{1}{3}+ \ . \ . \ . \ +\frac{1}{2n-1},$$

and
$$s_{2n}=\alpha_{2n}-\alpha_n.$$

Suppose we change the order, taking two positive terms and then one negative term. We have the series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+ \ . \ . \ . \ .$$

In s'_{3n} there are $2n$ positive terms and n negative

terms. Therefore

$$s'_{3n} = \left(\alpha_{4n} - \frac{\alpha_{2n}}{2}\right) - \frac{\alpha_n}{2}$$

$$= (\alpha_{4n} - \alpha_{2n}) + \frac{1}{2}(\alpha_{2n} - \alpha_n)$$

$$= s_{4n} + \frac{1}{2}s_{2n}.$$

Here are three sequences,

$$\begin{array}{llllll} & s_3' & s_6' \ \ldots & s'_{3n} \ \ldots & \text{representing} & S', \\ & s_4 & s_8 \ \ldots & s_{4n} \ \ldots & \text{``} & S, \\ \text{and} & \frac{1}{2}s_2 & \frac{1}{2}s_4 \ \ldots & \frac{1}{2}s_{2n} \ \ldots & \text{``} & \frac{1}{2}S, \end{array}$$

and the value of the new series is

$$S' = S + \frac{1}{2}S = \frac{3}{2}S. \quad \text{(See Arts. 41 and 44.)}$$

(Harkness and Morley)

If we take one positive and two negative terms we have the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ \ldots,$$

and we shall find $S' = \frac{1}{2}S$. In fact the series may be

written

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ \ldots. \qquad \text{(Laurent)}$$

One positive term taken with four negative terms will give a series

$$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\ \ldots$$

equal to zero.

Now, with the terms of the same series alternately positive and negative, change the order of the negative terms among themselves, taking first a fraction whose denominator is not divisible by 4 and then two fractions whose denominators are divisible by 4. This change produces the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{8}+\ \ldots.$$

and the sum of the first $6n$ terms is

$$s'_{6n}=\left(\alpha_{6n}-\frac{\alpha_{3n}}{2}\right)-\frac{1}{2}\left(\alpha_{2n}-\frac{\alpha_n}{2}\right)-\frac{1}{4}\alpha_{2n}$$

$$=(\alpha_{6n}-\alpha_{3n})+\frac{1}{2}(\alpha_{3n}-\alpha_{2n})-\frac{1}{4}(\alpha_{2n}-\alpha_n).$$

The first parenthesis is equal to s_{6n}. The rest may be

written

$$\frac{1}{4}\left[\frac{2}{2n+1}+\frac{2}{2n+2}+\ \cdots\ +\frac{2}{3n}\right.$$
$$\left.-\frac{1}{n+1}-\frac{1}{n+2}-\ \cdots\ -\frac{1}{2n}\right].$$

Every other term in the first line is cancelled by a term in the second line. Letting n be odd, there is left

$$\frac{1}{4}\left[\frac{2}{2n+1}+\frac{2}{2n+3}+\ \cdots\ +\frac{2}{3n}\right.$$
$$\left.-\frac{2}{3n+1}-\frac{2}{3n+3}-\ \cdots\ -\frac{2}{4n}\right]$$
$$=\frac{n}{2}\left[\frac{1}{(2n+1)(3n+1)}+\ \cdots\ +\frac{1}{3n.4n}\right].$$

Here the number of terms in the brackets is $\frac{n+1}{2}$. The first is the largest and the last the smallest. Therefore this expression is

$$<\frac{n(n+1)}{4(2n+1)(3n+1)}<\frac{n+1}{24n},$$

and
$$>\frac{n(n+1)}{4.3n.4n}>\frac{1}{48}.$$

It follows that

$$S' \overline{\overline{<}} S+\frac{1}{24} \quad\text{and}\quad \overline{\overline{>}} S+\frac{1}{48}. \qquad \text{(Art. 36)}$$

CHAPTER IV

POWER SERIES

I. The Radius of Convergence

84. A power series is a series whose terms contain as factors the successive powers of some number.

A power series is a series written in the form

$$u_1+u_2x+\ .\ .\ .\ +u_nx^{n-1}+\ .\ .\ .\ .$$

An example is the geometrical series

$$a+ax+\ .\ .\ .\ +ax^{n-1}+\ .\ .\ .\ .$$

This is absolutely convergent if x is any number numerically less than 1.

85. Theorem.—*If the series*

$$u_1+u_2x+\ .\ .\ .\ +u_nx^{n-1}\ .\ .\ .$$

is convergent for a particular value of x, it is absolutely convergent for any value of x numerically less.

Proof.—Let x_1 be a particular value of x for which the series is convergent, and let x be some number numerically less.

The geometrical series

$$1+\frac{x}{x_1}+\ . \ . \ . \ +\frac{x^{n-1}}{x_1{}^{n-1}}+\ . \ . \ .$$

is absolutely convergent, since $\frac{x}{x_1}$ is numerically less than 1. This series will still be absolutely convergent if we multiply its terms respectively by

$$u_1 \quad u_2x_1 \ . \ . \ . \ u_nx_1{}^{n-1} \ . \ . \ . \ ,$$

numbers which form a sequence equal to zero, since they are the terms of the given series when x has the particular value for which by hypothesis the series is convergent (Art. 76, Cor.).

Therefore the series

$$u_1+u_2x+\ . \ . \ . \ +u_nx^{n-1}+\ . \ . \ .$$

is absolutely convergent, x being numerically less than x_1.

Cor.—If a power series is divergent for a particular value of x, it is divergent for any value of x numerically greater.

For if there were some value of x numerically greater for which the given series was convergent, it would also be convergent for the given value of x, for which by hypothesis it is divergent.

86. Theorem.—*If a power series is convergent for some particular value of x and divergent for another value of x, there is a positive number r such that the series is abso-*

lutely convergent for all values of x numerically less than r and divergent for all values of x numerically greater than r.

Proof.—The series is always convergent or divergent, and if it is convergent for a value of x, a, and divergent for another value of x, b, then a is numerically less than b unless they are positive and negative numbers numerically equal. We can separate all numbers into two classes, putting into the first class all negative numbers and all positive numbers for which the series is convergent, and into the second class all positive numbers for which the series is divergent. The number determined by this separation is the number r.

The number r is called the radius of convergence.

If a series has a radius of convergence r it may be absolutely convergent or semi-convergent or divergent for $x=r$ or $x=-r$.

If a series has no radius of convergence it is either absolutely convergent for all values of x or divergent for all values of x except $x=0$.

For the geometrical series the radius of convergence is 1, and the series is divergent when $x=1$ and when $x=-1$.

For other examples see Art. 88.

87. Theorem.—*Given the series*

$$u_1+u_2x+\ .\ .\ .\ +u_nx^{n-1}+\ .\ .\ .\ ,$$

if the sequence

$$\frac{u_2}{u_1}\quad\frac{u_3}{u_2}\ .\ .\ .\ \frac{u_n}{u_{n-1}}\ .\ .\ .$$

represents a number a, then the series has a radius of convergence equal to the reciprocal of the numerical value of a.

Proof.—The ratios of the terms of the series, each to the preceding, form the sequence

$$\frac{u_2x}{u_1} \quad \frac{u_3x}{u_2} \quad \cdots \quad \frac{u_nx}{u_{n-1}} \quad \cdots$$

representing the number ax. This number is numerically less than 1, and the series is absolutely convergent, if

$$|x| < \frac{1}{|a|}.$$

On the other hand, if

$$|x| > \frac{1}{|a|}$$

the terms of the series will not form a sequence equal to zero and the series will not be convergent.

88. Examples.

(1) The series

$$1+x+\frac{x^2}{2}+ \quad \cdots \quad +\frac{x^n}{n}+ \quad \cdots$$

has 1 for radius of convergence. For $x=1$ the series is divergent, being the harmonic series; for $x=-1$ it is semi-convergent.

(2) The series

$$1+x+\frac{x^2}{\underline{|2}}+\ .\ .\ .\ +\frac{x^n}{\underline{|n}}+\ .\ .\ .$$

has no radius of convergence. It is absolutely convergent whatever the value of x. In fact, the ratio of the $(n+1)$th term to the preceding is $\frac{x}{n}$, and for different values of n this fraction will form a sequence equal to zero whatever the value of x.

(3) The series

$$1+x+\underline{|2}x^2+\ .\ .\ .\ +\underline{|n}x^n+\ .\ .\ .$$

has no radius of convergence. It is divergent for all values of x except zero.

(4) The series

$$1+c_1x+c_2x^2+\ .\ .\ .\ +c_nx^n+\ .\ .\ .\ ,$$

where $c_1=p$ and, in general,

$$c_n=\frac{p(p-1)\ .\ .\ .\ (p-n+1)}{\underline{|n}},$$

has 1 for radius of convergence except when p is zero or a positive integer. The ratio of coefficients is

$$\frac{c_n}{c_{n-1}}=\frac{p-n+1}{n}=-1+\frac{p+1}{n},$$

and for different values of n this forms a sequence equal to -1.

When $x=1$ or -1 the convergence depends on the value of p.

89. Theorem.—*If each of two power series has a radius of convergence or is convergent for all values of x, their product can be written as a power series; this will be absolutely convergent for any value of x which makes both of the given series absolutely convergent.*

Proof.—If the series are

$$u_1+u_2x+\ .\ .\ .\ +u_nx^{n-1}+\ .\ .\ .$$

and $$v_1+v_2x+\ .\ .\ .\ +v_nx^{n-1}+\ .\ .\ .\ ,$$

the product will be

$$u_1v_1+(u_1v_2+u_2v_1)x+\ .\ .\ .$$
$$+(u_1v_n+u_2v_{n-1}+\ .\ .\ .\ +u_nv_1)x^{n-1}+\ .\ .\ .$$
(Art. 82)

Any number of power series may be multiplied together if each has a radius of convergence or is convergent for all values of x, and the product series will be absolutely convergent for any value of x which makes all the given series absolutely convergent.

90. Theorem.—*If the series*

$$u_1+u_2x+\ .\ .\ .+u_nx^{n-1}+\ .\ .\ .$$

is convergent for a value of x whose numerical value is ρ,

then there is a positive number M such that for all values of n

$$|u_n| < \frac{M}{\rho^{n-1}}.$$

Proof.—For the given value of x the terms of the series form a sequence equal to zero and are all numerically less than a certain positive number M. Hence

$$|u_n|\rho^{n-1} < M,$$

or

$$|u_n| < \frac{M}{\rho^{n-1}}.$$

91. Theorem.—*Conversely, if there are two positive numbers M and ρ such that for all values of n*

$$|u_n| < \frac{M}{\rho^{n-1}},$$

then for any value of x numerically less than ρ the power series is absolutely convergent and has a value which is numerically less than

$$\frac{M\rho}{\rho - |x|}.$$

Proof.—The terms of the power series are numerically less than the corresponding terms of the geometrical series

$$M + \frac{M|x|}{\rho} + \ . \ . \ . \ + \frac{M|x|^{n-1}}{\rho^{n-1}} + \ . \ . \ . \ ,$$

which equals

$$\frac{M}{1-\frac{|x|}{\rho}}, \quad \text{or} \quad \frac{M\rho}{\rho-|x|},$$

when $|x|<\rho$. Therefore, when $|x|<\rho$, the given series is absolutely convergent and its value is numerically less than $\frac{M\rho}{\rho-|x|}$ (see Art. 68).

The series

$$M+\frac{Mx}{\rho}+\ .\ .\ .\ +\frac{Mx^{n-1}}{\rho^{n-1}}+\ .\ .\ .$$

is a series of comparison for the given series.

Cor.—If u_1 is zero, that is, if a power series begins with a term in x or some power of x, and if there are two positive numbers M and ρ such that for all values of n greater than 1 the coefficient of x^{n-1} is numerically less than $\frac{M}{\rho^{n-1}}$, then for values of x numerically less than ρ the value of the series is numerically less than

$$\frac{M|x|}{\rho-|x|}.$$

For the terms of the series are numerically less than the corresponding terms of the series

$$\frac{M|x|}{\rho}+\ .\ .\ .\ +\frac{M|x|^{n-1}}{\rho^{n-1}}+\ .\ .\ .\ ,$$

which equals

$$\frac{M|x|}{\rho-|x|}.$$

II. "Undetermined Coefficients"

92. Theorem.—*A power series having a radius of convergence or convergent for all values of x cannot equal zero for values of x forming a sequence equal to zero but not themselves zero.*

Proof.—As we may suppose some of the coefficients zero we will write the series

$$u_1x^p + \ . \ . \ . \ + u_nx^{p+n-1} + \ . \ . \ . \ ,$$

where p is zero or a positive integer, and where u_1 is not zero.

Write

$$P = u_1 + u_2x + \ . \ . \ . \ + u_nx^{n-1} + \ . \ . \ . \ ,$$

and $P' = u_2 + u_3x + \ . \ . \ . \ + u_{n+1}x^{n-1} + \ . \ . \ . \ .$

Both of these series are convergent for any value of x that makes the original series convergent, and

$$P = u_1 + xP'.$$

P is zero for any value of x except zero, which makes the original series zero, but P is not zero when $x=0$.

Since P' has a radius of convergence or is conver-

gent for all values of x, there are two positive numbers, M and ρ, such that for $|x|<\rho$

$$|xP'|<\frac{M|x|}{\rho-|x|}.$$

Therefore

$$|P|>|u_1|-\frac{M|x|}{\rho-|x|}.$$

ε being some positive number less than $|u_1|$, this expression will be greater than ε, that is,

$$|u_1|-\frac{M|x|}{\rho-|x|}>\varepsilon,$$

if

$$|x|<\frac{\rho(|u_1|-\varepsilon)}{M+|u_1|-\varepsilon}.$$

We will call this fraction ε'. ε' is less than ρ, since

$$\frac{|u_1|-\varepsilon}{M+|u_1|-\varepsilon}<1.$$

Then P is numerically greater than ε when x is numerically less than ε'. But in a sequence equal to zero there is a place beyond which all the elements are numerically less than ε', and for none of these numbers as values of x can P be zero, nor can the original series if these numbers are not themselves zero.

93. Theorem.—*If two series in powers of x, having radii of convergence or convergent for all values of x, represent*

the same numbers for values of x forming a sequence equal to zero and not themselves zero, then the two given series must be identically the same series, each coefficient of one equal to the corresponding coefficient of the other.

Proof.—If any of the corresponding coefficients were different we could, by subtracting one of the given series from the other, get a series equal to zero for values of x forming the given sequence, which is contrary to the last theorem.

Cor.—If two power series are convergent and equal for all values of x numerically less than a certain number not zero, they are equal term by term, each coefficient of one equal to the corresponding coefficient of the other.

For out of all values of x numerically less than the number we can select values to form a sequence equal to zero, and the two series, being equal for these values of x, have the same coefficients, by the theorem.

This form of the theorem is given in most of our Algebras as the "Theorem of undetermined coefficients."

94. Since a polynomial in x may be regarded as a power series with all the coefficients zero beyond a certain place, these theorems are true of such polynomials.

There is, however, a theorem concerning polynomials, proved in our Algebras in connection with the theory of equations, which is more general, and which we will state here for convenience of reference:

Theorem.—*A polynomial in x of degree n cannot equal zero for more than n different values of x.*

Cor.—If two polynomials of degree n are equal for more than n different values of x they are identically the same polynomial, equal for all values of x, each coefficient of one equal to the corresponding coefficient of the other.

CHAPTER V

THE EXPONENTIAL, BINOMIAL, AND LOGARITHMIC SERIES

I. THE EXPONENTIAL SERIES

95. We shall assume in this chapter that the binomial theorem has been proved for positive integer exponents.

In particular, r being any positive integer less than n, the $(r+1)$th term in the expansion of $(a+x)^n$ may be written

$$\frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}a^{n-r}x^r.$$

96. Theorem.—*x being any rational number,*

$$\left(1+1+\frac{1}{\underline{|2}}+\ \dots\ +\frac{1}{\underline{|n}}+\ \dots\right)^x$$

$$=1+x+\frac{x^2}{\underline{|2}}+\ \dots\ +\frac{x^n}{\underline{|n}}+\ \dots.$$

Proof.—Write

$$S_x=1+x+\frac{x^2}{\underline{|2}}+\ \dots\ +\frac{x^n}{\underline{|n}}+\ \dots$$

so that, y being any other number,

$$S_y = 1 + y + \frac{y^2}{\underline{|2}} + \ . \ . \ . \ + \frac{y^n}{\underline{|n}} + \ . \ . \ . \ .$$

The product of these two series is the series

$$1 + (x+y) + \left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) + \ . \ . \ .$$
$$+ \left(\frac{x^n}{\underline{|n}} + \frac{x^{n-1}y}{\underline{|n-1}\,\underline{|1}} + \ . \ . \ . \ + \frac{y^n}{\underline{|n}}\right) + \ . \ . \ .$$
$$= 1 + (x+y) + \frac{(x+y)^2}{\underline{|2}} + \ . \ . \ . \ + \frac{(x+y)^n}{\underline{|n}} + \ . \ . \ . \ .$$

That is, we may write it S_{x+y} and say

$$S_x S_y = S_{x+y}.$$

These series are convergent and this relation holds true for all values of x and y.

If we put $y = x,\ 2x, \ . \ . \ . \ rx, \ . \ . \ .$ we shall have

$$S_x^{\,2} = S_{2x},$$

and by induction $\qquad S_x^r = S_{rx},$

r being any positive integer.

In this formula put $\dfrac{rx}{s}$ for x and s for r:

$$S_{\frac{rx}{s}}^{\,s} = S_{rx} = S_x^r;$$

or, taking the sth root,

$$S_{\frac{r}{s}x}=S_x^{\frac{r}{s}}.$$

That is, for positive fractional values of r

$$S_x^r=S_{rx}.$$

If in the product series we put $y=-x$, it will reduce to its first term, 1; that is,

$$S_xS_{-x}=1,$$

or $$S_{-x}=\frac{1}{S_x}.$$

Now $$S_x^{-r}=\frac{1}{S_x^r}=\frac{1}{S_{rx}},$$

and from what we have just shown this is equal to S_{-rx}.

Hence, for negative rational values of r and, therefore, for all rational values of r,

$$S_x^r=S_{rx}.$$

If we put $x=1$ the series becomes

$$S=1+1+\frac{1}{\underline{|2}}+\ \ldots\ +\frac{1}{\underline{|n}}+\ \ldots\ .$$

This series represents a number which is denoted by the letter e. It is a positive number greater than 2.

Putting also x in place of r, we have for all rational values of x

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\ \ldots\ +\frac{x^n}{\underline{|n}}+\ \ldots\ .$$

97. Theorem.—*The same relation holds true for irrational values of x.*

Proof.—If x and y are any two numbers, rational or irrational, $x<y$, then $S_x<S_y$.

For, in the first place, if x and y are both positive, the terms in both series are positive and the terms in S_x are less than the corresponding terms in S_y, except the first, which is the same in both series. Again, if x and y are both negative, say $x=-x'$ and $y=-y'$, $x'>y'$, then

$$S_x=\frac{1}{S_{x'}} \quad \text{and} \quad S_y=\frac{1}{S_{y'}}.$$

$$S_{x'}>S_{y'} \quad \text{and, therefore,} \quad S_x<S_y.$$

Finally, if x is negative, say $x=-x'$, and y positive,

$$S_x=\frac{1}{S_{x'}}<1 \quad \text{and} \quad S_y>1.$$

Let a be an irrational number separating all rational numbers into two classes, and for the moment let x stand for any rational number in the first class and y any rational number in the second class. The number e^a occupies the point of separation of the numbers e^x and e^y into two classes and is determined by this separation

(Art. 24). The same numbers written in the form S_x and S_y determine the same point of separation, which is, therefore, occupied by the number S_a.

That is, writing now x in place of a, we have proved for irrational values of x, also,

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots .$$

II. The Binomial Theorem for any Rational Exponent

98. Theorem.—*p being any rational number and x any number numerically less than* 1,

$$(1+x)^p = 1 + {}_pc_1x + \ldots + {}_pc_nx^n + \ldots ,$$

where ${}_pc_1 = p$, and, in general,

$${}_pc_n = \frac{p(p-1)\ldots(p-n+1)}{\underline{|n}}.$$

Proof.—We assume that the theorem has already been proved for positive integer values of p.

Write

$$S_p = 1 + {}_pc_1x + \ldots + {}_pc_nx^n + \ldots ,$$

so that, q being some other number,

$$S_q = 1 + {}_qc_1x + \ldots + {}_qc_nx^n + \ldots .$$

The product of these two series is the series

$$1+({}_pc_1+{}_qc_1)x+\ \ldots$$
$$+({}_pc_n+{}_pc_{n-1}\,{}_qc_1+\ \ldots\ +{}_qc_n)x^n+\ \ldots\,.$$

When p and q are positive integers we know that this is the same as the series

$$1+{}_{p+q}c_1x+\ \ldots\ +{}_{p+q}c_nx^n+\ \ldots\,.$$

If p is a positive integer less than n, ${}_pc_n=0$; and if q is a positive integer less than n, ${}_qc_n=0$. But even for such values of p and q, as well as for all integer values greater than n, we have the relation

$${}_pc_n+{}_pc_{n-1}\,{}_qc_1+\ \ldots\ +{}_qc_n={}_{p+q}c_n,$$

n being any positive integer.

The two members of this equation are polynomials of degree n in p and q. For any positive integer values of p and q they are equal. If we put for q any positive integer they become polynomials of degree n in p, equal for all positive integer values of p; that is, equal for more than n values of p, and, therefore, for all values of p.

The two members of this equation are, then, equal for any positive integer value of q combined with any value whatever of p. If we put for p any value whatever they become polynomials of degree n in q, equal for all positive integer values of q; that is, equal for more than n values of q, and, therefore, for all values of q.

That is, for any value whatever of p and any value whatever of q we have

$$S_p S_q = S_{p+q}.$$

Putting $q=p$, $2p$, etc., and proceeding as in the case of the exponential series, we prove for every positive or negative rational number r the relation

$$S_p^r = S_{rp}.$$

For $p=1$ the relation becomes

$$S_1 = 1+x.$$

Therefore, putting also p in place of r, we have for all rational values of p, x being numerically less than 1,

$$(1+x)^p = 1 + {}_pc_1 x + \ . \ . \ . \ + {}_pc_n x^n + \ . \ . \ . \ .$$

This proof and the proof of the last section for the series representing e^x are due in part to Euler.

III. The Binomial Theorem for an Irrational Exponent and the Logarithmic Series

99. In what follows we shall take e for the base of all logarithms.

If $1+x=e^\mu$, so that $\mu=\log(1+x)$, then the formula

$$e^{\mu y} = 1 + \mu y + \ . \ . \ . \ + \frac{\mu^n y^n}{\underline{|n}} + \ . \ . \ .$$

becomes

$$(1) \qquad (1+x)^y = 1 + \{\log(1+x)\}y + \dots$$

$$+ \{\log(1+x)\}^n \frac{y^n}{\underline{|n}} + \dots.$$

But if x is numerically less than 1, and y is a rational number, we have also

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \dots$$

$$+ \frac{y(y-1)\dots(y-n+1)}{\underline{|n}}x^n + \dots$$

$$= 1 + yx + \left(\frac{y^2x^2}{\underline{|2}} - \frac{yx^2}{\underline{|2}}\right) + \dots$$

$$+ \left(\frac{y^nx^n}{\underline{|n}} - \dots + (-1)^{n-1}\frac{\underline{|n-1}}{\underline{|n}}yx^n\right) + \dots.$$

This series is convergent when x is numerically less than 1, whatever the value of y. Now in each parenthesis the terms of even degree in x and y taken together have plus signs and the terms of odd degree have minus signs. If we put for x and y the negative numbers which have, respectively, the same numerical values, we shall be putting for these terms their numerical values. x being numerically less than 1, the series will be convergent and the parentheses may be removed (Art. 73 (1)).

That is, whatever the values of x and y, $|x|<1$, the parentheses in the above series may be removed and the series as then written will be absolutely convergent, namely,

$$(\alpha) \qquad 1+yx+\frac{y^2x^2}{\underline{|2}}-\frac{yx^2}{2}+\dots \\ +\frac{y^nx^n}{\underline{|n}}-\dots+(-1)^{n-1}\frac{yx^n}{n}+\dots;$$

representing $(1+x)^y$ when y is also rational.

If from the series (α) we select all the terms that contain y^n, we have a series from which we can take the factor y^n, say u_ny^n, u_n being a power series in x convergent for values of x numerically less than 1 (Art. 67). The first term of u_n is $\frac{x^n}{\underline{|n}}$, and its coefficients are alternately positive and negative.

In particular,

$$u_1=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n-1}\frac{x^n}{n}+\dots.$$

Consider the series

$$(\beta) \qquad 1+u_1y+\dots+u_ny^n+\dots.$$

The terms of this series are numbers represented by part-series taken from the series (α). If in the series (α) we replace the terms by their numerical values. we

have the convergent series of positive terms

$$(\alpha') \qquad 1+|yx|+\frac{|y^2x^2|}{\underline{|2}}+\frac{|yx^2|}{2}+\ \ldots \\ +\frac{|y^nx^n|}{\underline{|n}}+\ \ldots\ +\frac{|yx^n|}{n}+\ \ldots ;$$

and corresponding to the series (β) the series

$$(\beta') \qquad 1+u_1'|y|+\ \ldots\ +u_n'|y^n|+\ \ldots ,$$

where u_n' is the series of numerical values of the terms of u_n.

The sum of the first n terms of (β') is the sum of a certain number of part-series from (α'), and as a whole may be regarded as a part-series taken from (α'). Its value is less than the value of (α'), and therefore the series (β') is convergent and its value is equal to or less than the value of the series (α').

But all the terms of (α') are found in the different series which represent the terms of (β'), and therefore the sum of any number of terms of (α') is less than the sum of a certain number of terms of (β'). That is, the value of the series (α') is equal to or less than the value of the series (β').

It follows that the two series must have the same value.

Since $|u_n| \overline{\gtrless} u_n'$ (Art. 62, Cor.), the series (β) is absolutely convergent whenever the series (β') is convergent.

If we take from the series (α') the part-series corresponding to the sum of the first n terms of the series

(β') we shall have a remainder series R_n' of positive terms whose value will be less than ϵ when n is taken sufficiently large (Art. 67, Cor.).

If in the same way we take from the series (α) the part-series corresponding to the sum of the first n terms of the series (β) we shall have a remainder series R_n whose terms have for numerical values the corresponding terms of the series R_n'. Therefore the numerical value of R_n will also be less than ϵ, and the sum of the first n terms of the series (β) will differ from the value of the series (α) by less than ϵ.

This proves that the series (β) has the same value as the series (α).

Now for rational values of y, x being numerically less than 1, the series (α) represents $(1+x)^y$, which is also represented by the series (1). That is, for any value of x numerically less than 1 the series (β) and the series (1) are convergent power series in y, equal for all rational values of y.

But from all rational values of y we can select a sequence equal to zero. Therefore, by the theorem of Art. 93, the two series (β) and (1) are identical, each coefficient of one equal to the corresponding coefficient of the other.

100. Since the series (α) has the same value as the series (β) for all values of y, $|x|<1$, and since the series (β) is identical with the series (1), which for all values of y represents $(1+x)^y$, it follows that the series (α) represents $(1+x)^y$ for all values of y, $|x|<1$.

Thus we prove the binomial theorem for irrational values of the exponent; namely, writing p for y,

Theorem.—*For irrational values of p, x being numerically less than* 1,

$$(1+x)^p = 1 + {}_pc_1x + \ . \ . \ . \ + {}_pc_nx^n + \ . \ . \ . \ .$$

101. The coefficient of y in (1) is $\log(1+x)$; therefore we have the following theorem:

Theorem.—*For all values of x numerically less than* 1,

$$\log(1+x) = x - \frac{x^2}{2} + \ . \ . \ . \ + (-1)^{n-1}\frac{x^n}{n} + \ . \ . \ . \ .$$

SHORT-TITLE CATALOGUE

OF THE

PUBLICATIONS

OF

JOHN WILEY & SONS,

NEW YORK.

LONDON: CHAPMAN & HALL, LIMITED.

ARRANGED UNDER SUBJECTS.

Descriptive circulars sent on application. Books marked with an asterisk are sold at *net* prices only, a double asterisk (**) books sold under the rules of the American Publishers' Association at *net* prices subject to an extra charge for postage. All books are bound in cloth unless otherwise stated.

AGRICULTURE.

Armsby's Manual of Cattle-feeding	12mo,	$1 75
Principles of Animal Nutrition	8vo,	4 00
Budd and Hansen's American Horticultural Manual:		
Part I.—Propagation, Culture, and Improvement	12mo,	1 50
Part II.—Systematic Pomology	12mo,	1 50
Downing's Fruits and Fruit-trees of America	8vo,	5 00
Elliott's Engineering for Land Drainage	12mo,	1 50
Practical Farm Drainage	12mo,	1 00
Green's Principles of American Forestry	12mo,	1 50
Grotenfelt's Principles of Modern Dairy Practice. (Woll.)	12mo,	2 00
Kemp's Landscape Gardening	12mo,	2 50
Maynard's Landscape Gardening as Applied to Home Decoration	12mo,	1 50
Sanderson's Insects Injurious to Staple Crops	12mo,	1 50
Insects Injurious to Garden Crops. (*In preparation.*)		
Insects Injuring Fruits. (*In preparation.*)		
Stockbridge's Rocks and Soils	8vo,	2 50
Woll's Handbook for Farmers and Dairymen	16mo,	1 50

ARCHITECTURE.

Baldwin's Steam Heating for Buildings	12mo,	2 50
Berg's Buildings and Structures of American Railroads	4to,	5 00
Birkmire's Planning and Construction of American Theatres	8vo,	3 00
Architectural Iron and Steel	8vo,	3 50
Compound Riveted Girders as Applied in Buildings	8vo,	2 00
Planning and Construction of High Office Buildings	8vo,	3 50
Skeleton Construction in Buildings	8vo,	3 00
Briggs's Modern American School Buildings	8vo,	4 00
Carpenter's Heating and Ventilating of Buildings	8vo,	4 00
Freitag's Architectural Engineering. 2d Edition, Rewritten	8vo,	3 50
Fireproofing of Steel Buildings	8vo,	2 50
French and Ives's Stereotomy	8vo,	2 50
Gerhard's Guide to Sanitary House-inspection	16mo,	1 00
Theatre Fires and Panics	12mo,	1 50
Holly's Carpenters' and Joiners' Handbook	18mo,	75
Johnson's Statics by Algebraic and Graphic Methods	8vo,	2 00

Kidder's Architect's and Builder's Pocket-book. Rewritten Edition. 16mo, mor., 5 00
Merrill's Stones for Building and Decoration..........................8vo, 5 00
Non-metallic Minerals: Their Occurrence and Uses.............8vo, 4 00
Monckton's Stair-building 4to, 4 00
Patton's Practical Treatise on Foundations............................8vo, 5 00
Peabody's Naval Architecture..8vo, 7 50
Richey's Handbook for Superintendents of Construction. (*In press.*)
Sabin's Industrial and Artistic Technology of Paints and Varnish......8vo, 3 00
Siebert and Biggin's Modern Stone-cutting and Masonry................8vo, 1 50
Snow's Principal Species of Wood......................................8vo, 3 50
Sondericker's Graphic Statics with Applications to Trusses, Beams, and Arches. 8vo, 2 00
Towne's Locks and Builders' Hardware....................18mo, morocco, 3 00
Wait's Engineering and Architectural Jurisprudence....................8vo, 6 00
Sheep, 6 50
Law of Operations Preliminary to Construction in Engineering and Architecture..8vo, 5 00
Sheep, 5 50
Law of Contracts..8vo, 3 00
Wood's Rustless Coatings: Corrosion and Electrolysis of Iron and Steel...8vo, 4 00
Woodbury's Fire Protection of Mills...................................8vo, 2 50
Worcester and Atkinson's Small Hospitals, Establishment and Maintenance, Suggestions for Hospital Architecture, with Plans for a Small Hospital. 12mo, 1 25
The World's Columbian Exposition of 1893....................Large 4to, 1 00

ARMY AND NAVY.

Bernadou's Smokeless Powder, Nitro-cellulose, and the Theory of the Cellulose Molecule..12mo, 2 50
* Bruff's Text-book Ordnance and Gunnery..............................8vo, 6 00
Chase's Screw Propellers and Marine Propulsion........................8vo, 3 00
Craig's Azimuth...4to, 3 50
Crehore and Squire's Polarizing Photo-chronograph.....................8vo, 3 00
Cronkhite's Gunnery for Non-commissioned Officers.........24mo, morocco, 2 00
* Davis's Elements of Law...8vo, 2 50
* Treatise on the Military Law of United States..................8vo, 7 00
Sheep, 7 50
De Brack's Cavalry Outpost Duties. (Carr.)................24mo morocco, 2 00
Dietz's Soldier's First Aid Handbook......................16mo, morocco, 1 25
* Dredge's Modern French Artillery.....................4to, half morocco, 15 00
Durand's Resistance and Propulsion of Ships...........................8vo, 5 00
* Dyer's Handbook of Light Artillery..................................12mo, 3 00
Eissler's Modern High Explosives......................................8vo, 4 00
* Fiebeger's Text-book on Field Fortification....................Small 8vo, 2 00
Hamilton's The Gunner's Catechism....................................18mo, 1 00
* Hoff's Elementary Naval Tactics.....................................8vo, 1 50
Ingalls's Handbook of Problems in Direct Fire.........................8vo, 4 00
* Ballistic Tables...8vo, 1 50
* Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II..8vo. each, 6 00
* Mahan's Permanent Fortifications. (Mercur.)........8vo, half morocco, 7 50
Manual for Courts-martial.................................16mo, morocco, 1 50
* Mercur's Attack of Fortified Places.................................12mo, 2 00
* Elements of the Art of War...8vo, 4 00
Metcalf's Cost of Manufactures—And the Administration of Workshops, Public and Private...8vo, 5 00
* Ordnance and Gunnery. 2 vols......................................12mo, 5 00
Murray's Infantry Drill Regulations.......................18mo, paper, 10
Peabody's Naval Architecture..8vo 7 50

* Phelps's Practical Marine Surveying..............................8vo, 2 50
Powell's Army Officer's Examiner..............................12mo, 4 00
Sharpe's Art of Subsisting Armies in War...................18mo, morocco, 1 50
* Walke's Lectures on Explosives..............................8vo, 4 00
* Wheeler's Siege Operations and Military Mining..............................8vo, 2 00
Winthrop's Abridgment of Military Law..............................12mo, 2 50
Woodhull's Notes on Military Hygiene..............................16mo, 1 50
Young's Simple Elements of Navigation...................16mo morocco, 1 00
Second Edition, Enlarged and Revised...................16mo, morocco, 2 00

ASSAYING.

Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe. 12mo, morocco, 1 50
Furman's Manual of Practical Assaying..............................8vo, 3 00
Lodge's Notes on Assaying and Metallurgical Laboratory Experiments....8vo, 3 00
Miller's Manual of Assaying..............................12mo, 1 00
O'Driscoll's Notes on the Treatment of Gold Ores..............................8vo, 2 00
Ricketts and Miller's Notes on Assaying..............................8vo, 3 00
Ulke's Modern Electrolytic Copper Refining..............................8vo, 3 00
Wilson's Cyanide Processes..............................12mo, 1 50
Chlorination Process..............................12mo, 1 50

ASTRONOMY.

Comstock's Field Astronomy for Engineers..............................8vo, 2 50
Craig's Azimuth..............................4to, 3 50
Doolittle's Treatise on Practical Astronomy..............................8vo, 4 00
Gore's Elements of Geodesy..............................8vo, 2 50
Hayford's Text-book of Geodetic Astronomy..............................8vo, 3 00
Merriman's Elements of Precise Surveying and Geodesy..............................8vo, 2 50
* Michie and Harlow's Practical Astronomy..............................8vo, 3 00
* White's Elements of Theoretical and Descriptive Astronomy.........12mo, 2 00

BOTANY.

Davenport's Statistical Methods, with Special Reference to Biological Variation. 16mo, morocco, 1 25
Thomé and Bennett's Structural and Physiological Botany..............16mo, 2 25
Westermaier's Compendium of General Botany. (Schneider.)..........8vo, 2 00

CHEMISTRY.

Adriance's Laboratory Calculations and Specific Gravity Tables.........12mo, 1 25
Allen's Tables for Iron Analysis..............................8vo, 3 00
Arnold's Compendium of Chemistry. (Mandel.)..................Small 8vo, 3 50
Austen's Notes for Chemical Students..............................12mo, 1 50
* Austen and Langworthy. The Occurrence of Aluminium in Vegetable Products, Animal Products, and Natural Waters..............8vo, 2 00
Bernadou's Smokeless Powder.—Nitro-cellulose, and Theory of the Cellulose Molecule..............................12mo, 2 50
Bolton's Quantitative Analysis..............................8vo, 1 50
* Browning's Introduction to the Rarer Elements 8vo, 1 50
Brush and Penfield's Manual of Determinative Mineralogy..............8vo, 4 00
Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.) 8vo, 3 00
Cohn's Indicators and Test-papers..............................12mo, 2 00
Tests and Reagents..............................8vo, 3 00
Craft's Short Course in Qualitative Chemical Analysis. (Schaeffer.)....12mo, 1 50
Dolezalek's Theory of the Lead Accumulator (Storage Battery). (Von Ende)..............................12mo, 2 50
Drechsel's Chemical Reactions. (Merrill.)..............................12mo, 1 25
Duhem's Thermodynamics and Chemistry. (Burgess.) 8vo, 4 00
Eissler's Modern High Explosives..............................8vo, 4 00
Effront's Enzymes and their Applications. (Prescott.)..............................8vo, 3 00

Erdmann's Introduction to Chemical Preparations. (Dunlap.)........12mo, 1 25
Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe 12mo, morocco, 1 50
Fowler's Sewage Works Analyses................................12mo, 2 00
Fresenius's Manual of Qualitative Chemical Analysis. (Wells.).........8vo, 5 00
Manual of Qualitative Chemical Analysis. Part I. Descriptive. (Wells.) 8vo, 3 00
System of Instruction in Quantitative Chemical Analysis. (Cohn.) 2 vols...8vo, 12 50
Fuertes's Water and Public Health................................12mo, 1 50
Furman's Manual of Practical Assaying............................8vo, 3 00
Getman's Exercises in Physical Chemistry.........................12mo,
Gill's Gas and Fuel Analysis for Engineers.......................12mo, 1 25
Grotenfelt's Principles of Modern Dairy Practice. (Woll.).............12mo, 2 00
Hammarsten's Text-book of Physiological Chemistry. (Mandel.).......8vo, 4 00
Helm's Principles of Mathematical Chemistry. (Morgan.)............12mo, 1 50
Hering's Ready Reference Tables (Conversion Factors).......16mo, morocco, 2 50
Hinds's Inorganic Chemistry......................................8vo, 3 00
* Laboratory Manual for Students...............................12mo, 75
Holleman's Text-book of Inorganic Chemistry. (Cooper.).............8vo, 2 50
Text-book of Organic Chemistry. (Walker and Mott.)............8vo, 2 50
* Laboratory Manual of Organic Chemistry. (Walker.)............12mo, 1 00
Hopkins's Oil-chemists' Handbook.................................8vo, 3 00
Jackson's Directions for Laboratory Work in Physiological Chemistry..8vo, 1 25
Keep's Cast Iron...8vo, 2 50
Ladd's Manual of Quantitative Chemical Analysis...................12mo, 1 00
Landauer's Spectrum Analysis. (Tingle.)...........................8vo, 3 00
Lassar-Cohn's Practical Urinary Analysis. (Lorenz.). 12mo, 1 00
Application of Some General Reactions to Investigations in Organic Chemistry. (Tingle.).......................................12mo, 1 00
Leach's The Inspection and Analysis of Food with Special Reference to State Control...8vo, 7 50
Löb's Electrolysis and Electrosynthesis of Organic Compounds. (Lorenz.) 12mo, 1 00
Lodge's Notes on Assaying and Metallurgical Laboratory Experiments....8vo, 3 00
Lunge's Techno-chemical Analysis. (Cohn.).......................12mo, 1 00
Mandel's Handbook for Bio-chemical Laboratory....................12mo, 1 50
* Martin's Laboratory Guide to Qualitative Analysis with the Blowpipe..12mo, 60
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten.................................8vo, 4 00
Examination of Water. (Chemical and Bacteriological.).........12mo, 1 25
Matthews's The Textile Fibres.....................................8vo, 3 50
Meyer's Determination of Radicles in Carbon Compounds. (Tingle.)..12mo, 1 00
Miller's Manual of Assaying......................................12mo, 1 00
Mixter's Elementary Text-book of Chemistry.......................12mo, 1 50
Morgan's Outline of Theory of Solution and its Results.................12mo, 1 00
Elements of Physical Chemistry..............................12mo, 2 00
Morse's Calculations used in Cane-sugar Factories..........16mo, morocco, 1 50
Mulliken's General Method for the Identification of Pure Organic Compounds. Vol. I. ....Large 8vo, 5 00
O'Brine's Laboratory Guide in Chemical Analysis......................8vo, 2 00
O'Driscoll's Notes on the Treatment of Gold Ores.....................8vo, 2 00
Ostwald's Conversations on Chemistry. Part One. (Ramsey.) (*In press.*)
* Penfield's Notes on Determinative Mineralogy and Record of Mineral Tests. 8vo, paper, 50
Pictet's The Alkaloids and their Chemical Constitution. (Biddle.).......8vo, 5 00
Pinner's Introduction to Organic Chemistry. (Austen.)..............12mo, 1 50
Poole's Calorific Power of Fuels...................................8vo, 3 00
Prescott and Winslow's Elements of Water Bacteriology, with Special Reference to Sanitary Water Analysis.........................12mo, 1 25

* Reisig's Guide to Piece-dyeing....................................8vo, 25 00
Richards and Woodman's Air, Water, and Food from a Sanitary Standpoint. 8vo, 2 00
Richards's Cost of Living as Modified by Sanitary Science..............12mo. 1 00
Cost of Food a Study in Dietaries..............................12mo, 1 00
* Richards and Williams's The Dietary Computer.......................8vo, 1 50
Ricketts and Russell's Skeleton Notes upon Inorganic Chemistry. (Part I.—Non-metallic Elements.)........................8vo, morocco, 75
Ricketts and Miller's Notes on Assaying..............................8vo, 3 00
Rideal's Sewage and the Bacterial Purification of Sewage................8vo, 3 50
Disinfection and the Preservation of Food........................8vo, 4 00
Riggs's Elementary Manual for the Chemical Laboratory.............8vo, 1 25
Rostoski's Serum Diagnosis. (Bolduan.)..........................12mo, 1 00
Ruddiman's Incompatibilities in Prescriptions........................8vo, 2 00
Sabin's Industrial and Artistic Technology of Paints and Varnish........8vo, 3 00
Salkowski's Physiological and Pathological Chemistry. (Orndorff.)....8vo, 2 50
Schimpf's Text-book of Volumetric Analysis........................12mo, 2 50
Essentials of Volumetric Analysis..............................12mo, 1 25
Spencer's Handbook for Chemists of Beet-sugar Houses......16mo, morocco, 3 00
Handbook for Sugar Manufacturers and their Chemists..16mo, morocco, 2 00
Stockbridge's Rocks and Soils......................................8vo, 2 50
* Tillman's Elementary Lessons in Heat...........................8vo, 1 50
* Descriptive General Chemistry................................8vo, 3 00
Treadwell's Qualitative Analysis. (Hall.).............................8vo, 3 00
Quantitative Analysis. (Hall.)....................................8vo, 4 00
Turneaure and Russell's Public Water-supplies.......................8vo, 5 00
Van Deventer's Physical Chemistry for Beginners. (Boltwood.).......12mo, 1 50
* Walke's Lectures on Explosives......................................8vo, 4 00
Washington's Manual of the Chemical Analysis of Rocks..............8vo, 2 00
Wassermann's Immune Sera: Hæmolysins, Cytotoxins, and Precipitins. (Bolduan.)..12mo, 1 00
Wells's Laboratory Guide in Qualitative Chemical Analysis..............8vo, 1 50
Short Course in Inorganic Qualitative Chemical Analysis for Engineering Students..12mo, 1 50
Whipple's Microscopy of Drinking-water...............................8vo, 3 50
Wiechmann's Sugar Analysis..............................Small 8vo, 2 50
Wilson's Cyanide Processes..12mo, 1 50
Chlorination Process..12mo, 1 50
Wulling's Elementary Course in Inorganic Pharmaceutical and Medical Chemistry...12mo, 2 00

CIVIL ENGINEERING.

BRIDGES AND ROOFS. HYDRAULICS. MATERIALS OF ENGINEERING RAILWAY ENGINEERING.

Baker's Engineers' Surveying Instruments..........................12mo, 3 00
Bixby's Graphical Computing Table...............Paper $19\frac{1}{2} \times 24\frac{1}{4}$ inches. 25
** Burr's Ancient and Modern Engineering and the Isthmian Canal. (Postage, 27 cents additional.)..................................8vo, net, 3 50
Comstock's Field Astronomy for Engineers...........................8vo, 2 50
Davis's Elevation and Stadia Tables.................................8vo, 1 00
Elliott's Engineering for Land Drainage.............................12mo, 1 50
Practical Farm Drainage...12mo, 1 00
Folwell's Sewerage. (Designing and Maintenance.)...................8vo, 3 00
Freitag's Architectural Engineering. 2d Edition Rewritten.............8vo 3 50
French and Ives's Stereotomy..8vo, 2 50
Goodhue's Municipal Improvements................................12mo, 1 75
Goodrich's Economic Disposal of Towns' Refuse.......................8vo, 3 50
Gore's Elements of Geodesy...8vo, 2 50
Hayford's Text-book of Geodetic Astronomy...........................8vo, 3 00
Hering's Ready Reference Tables (Conversion Factors).......16mo, morocco, 2 50

Howe's Retaining Walls for Earth..12mo, 1 25
Johnson's (J. B.) Theory and Practice of Surveying.............Small 8vo, 4 00
Johnson's (L. J.) Statics by Algebraic and Graphic Methods.............8vo, 2 00
Laplace's Philosophical Essay on Probabilities. (Truscott and Emory.) 12mo, 2 00
Mahan's Treatise on Civil Engineering. (1873.) (Wood.).............8vo, 5 00
* Descriptive Geometry..8vo, 1 50
Merriman's Elements of Precise Surveying and Geodesy.................8vo, 2 50
Elements of Sanitary Engineering..................................8vo, 2 00
Merriman and Brooks's Handbook for Surveyors..............16mo, morocco, 2 00
Nugent's Plane Surveying..8vo 3 50
Ogden's Sewer Design..12mo, 2 00
Patton's Treatise on Civil Engineering....................8vo half leather, 7 50
Reed's Topographical Drawing and Sketching..........................4to, 5 00
Rideal's Sewage and the Bacterial Purification of Sewage.................8vo, 3 50
Siebert and Biggin's Modern Stone-cutting and Masonry.................8vo, 1 50
Smith's Manual of Topographical Drawing. (McMillan.)..............8vo, 2 50
Sondericker's Graphic Statics, with Applications to Trusses, Beams, and Arches...8vo, 2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. (*In press.*)
* Trautwine's Civil Engineer's Pocket-book................16mo, morocco, 5 00
Wait's Engineering and Architectural Jurisprudence....................8vo, 6 00
Sheep, 6 50
Law of Operations Preliminary to Construction in Engineering and Architecture..8vo, 5 00
Sheep, 5 50
Law of Contracts..8vo, 3 00
Warren's Stereotomy—Problems in Stone-cutting.....................8vo, 2 50
Webb's Problems in the Use and Adjustment of Engineering Instruments. 16mo, morocco, 1 25
* Wheeler's Elementary Course of Civil Engineering...................8vo, 4 00
Wilson's Topographic Surveying...8vo, 3 50

BRIDGES AND ROOFS.

Boller's Practical Treatise on the Construction of Iron Highway Bridges..8vo, 2 00
* Thames River Bridge.................................4to, paper, 5 00
Burr's Course on the Stresses in Bridges and Roof Trusses, Arched Ribs, and Suspension Bridges..8vo, 3 50
Du Bois's Mechanics of Engineering. Vol. II.................Small 4to, 10 00
Foster's Treatise on Wooden Trestle Bridges..........................4to, 5 00
Fowler's Coffer-dam Process for Piers...................................8vo, 2 50
Ordinary Foundations..8vo, 3 50
Greene's Roof Trusses..8vo, 1 25
Bridge Trusses...8vo, 2 50
Arches in Wood, Iron, and Stone...8vo, 2 50
Howe's Treatise on Arches..8vo, 4 00
Design of Simple Roof-trusses in Wood and Steel.....................8vo, 2 00
Johnson, Bryan, and Turneaure's Theory and Practice in the Designing of Modern Framed Structures.............................Small 4to, 10 00
Merriman and Jacoby's Text-book on Roofs and Bridges:
Part I.—Stresses in Simple Trusses....................................8vo, 2 50
Part II.—Graphic Statics...8vo, 2 50
Part III.—Bridge Design. 4th Edition, Rewritten.....................8vo, 2 50
Part IV.—Higher Structures..8vo, 2 50
Morison's Memphis Bridge..4to, 10 00
Waddell's De Pontibus, a Pocket-book for Bridge Engineers...16mo, morocco, 3 00
Specifications for Steel Bridges.......................................12mo, 1 25
Wood's Treatise on the Theory of the Construction of Bridges and Roofs.8vo, 2 00
Wright's Designing of Draw-spans:
Part I.—Plate-girder Draws...8vo, 2 50
Part II.—Riveted-truss and Pin-connected Long-span Draws.......8vo, 2 50
Two parts in one volume...8vo, 3 50

HYDRAULICS.

Bazin's Experiments upon the Contraction of the Liquid Vein Issuing from an Orifice. (Trautwine.) 8vo, 2 00
Bovey's Treatise on Hydraulics........8vo, 5 00
Church's Mechanics of Engineering........8vo, 6 00
Diagrams of Mean Velocity of Water in Open Channels........paper, 1 50
Coffin's Graphical Solution of Hydraulic Problems........16mo, morocco, 2 50
Flather's Dynamometers, and the Measurement of Power........12mo, 3 00
Folwell's Water-supply Engineering........8vo, 4 00
Frizell's Water-power........8vo, 5 00
Fuertes's Water and Public Health........12mo, 1 50
Water-filtration Works........12mo, 2 50
Ganguillet and Kutter's General Formula for the Uniform Flow of Water in Rivers and Other Channels. (Hering and Trautwine.)........8vo, 4 00
Hazen's Filtration of Public Water-supply........8vo, 3 00
Hazlehurst's Towers and Tanks for Water-works........8vo, 2 50
Herschel's 115 Experiments on the Carrying Capacity of Large, Riveted, Metal Conduits........8vo, 2 00
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten........8vo, 4 00
Merriman's Treatise on Hydraulics. 9th Edition, Rewritten........8vo, 5 00
*** Michie's Elements of Analytical Mechanics**........8vo, 4 00
Schuyler's Reservoirs for Irrigation, Water-power, and Domestic Water-supply........Large 8vo, 5 00
**** Thomas and Watt's Improvement of Rivers. (Post., 44 c. additional), 4to,** 6 00
Turneaure and Russell's Public Water-supplies........8vo, 5 00
Wegmann's Design and Construction of Dams........4to, 5 00
Water-supply of the City of New York from 1658 to 1895........4to, 10 00
Weisbach's Hydraulics and Hydraulic Motors. (Du Bois.)........8vo, 5 00
Wilson's Manual of Irrigation Engineering........Small 8vo, 4 00
Wolff's Windmill as a Prime Mover........8vo, 3 00
Wood's Turbines........8vo, 2 50
Elements of Analytical Mechanics........8vo, 3 00

MATERIALS OF ENGINEERING.

Baker's Treatise on Masonry Construction........8vo, 5 00
Roads and Pavements........8vo, 5 00
Black's United States Public Works........Oblong 4to, 5 00
Bovey's Strength of Materials and Theory of Structures........8vo, 7 50
Burr's Elasticity and Resistance of the Materials of Engineering. 6th Edition, Rewritten........8vo, 7 50
Byrne's Highway Construction........8vo, 5 00
Inspection of the Materials and Workmanship Employed in Construction. 16mo, 3 00
Church's Mechanics of Engineering........8vo, 6 00
Du Bois's Mechanics of Engineering. Vol. I........Small 4to, 7 50
Johnson's Materials of Construction........Large 8vo, 6 00
Fowler's Ordinary Foundations........8vo, 3 50
Keep's Cast Iron........8vo, 2 50
Lanza's Applied Mechanics........8vo, 7 50
Martens's Handbook on Testing Materials. (Henning.) 2 vols........8vo, 7 50
Merrill's Stones for Building and Decoration........8vo, 5 00
Merriman's Text-book on the Mechanics of Materials........8vo, 4 00
Strength of Materials........12mo, 1 00
Metcalf's Steel. A Manual for Steel-users........12mo, 2 00
Patton's Practical Treatise on Foundations........8vo, 5 00
Richey's Handbook for Building Superintendents of Construction. (*In press.*)
Rockwell's Roads and Pavements in France........12mo, 1 25

Sabin's Industrial and Artistic Technology of Paints and Varnish......8vo, 3 00
Smith's Materials of Machines..12mo, 1 00
Snow's Principal Species of Wood....................................8vo, 3 50
Spalding's Hydraulic Cement..12mo, 2 00
Text-book on Roads and Pavements..................................12mo, 2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. (*In press.*)
Thurston's Materials of Engineering. 3 Parts.........................8vo, 8 00
Part I.—Non-metallic Materials of Engineering and Metallurgy.....8vo, 2 00
Part II.—Iron and Steel...8vo, 3 50
Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents...8vo, 2 50
Thurston's Text-book of the Materials of Construction..................8vo, 5 00
Tillson's Street Pavements and Paving Materials.......................8vo, 4 00
Waddell's De Pontibus. (A Pocket-book for Bridge Engineers.)..16mo, mor., 3 00
Specifications for Steel Bridges...12mo, 1 25
Wood's (De V.) Treatise on the Resistance of Materials, and an Appendix on the Preservation of Timber...8vo, 2 00
Wood's (De V.) Elements of Analytical Mechanics.......................8vo, 3 00
Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel...8vo, 4 00

RAILWAY ENGINEERING.

Andrews's Handbook for Street Railway Engineers.....3x5 inches, morocco, 1 25
Berg's Buildings and Structures of American Railroads..................4to, 5 00
Brooks's Handbook of Street Railroad Location.............16mo, morocco, 1 50
Butts's Civil Engineer's Field-book.........................16mo, morocco, 2 50
Crandall's Transition Curve..................................16mo, morocco, 1 50
Railway and Other Earthwork Tables.......................................8vo, 1 50
Dawson's "Engineering" and Electric Traction Pocket-book. 16mo, morocco, 5 00
Dredge's History of the Pennsylvania Railroad: (1879)..............Paper, 5 00
* **Drinker's Tunneling, Explosive Compounds, and Rock Drills,** 4to, half mor., 25 00
Fisher's Table of Cubic Yards...................................Cardboard, 25
Godwin's Railroad Engineers' Field-book and Explorers' Guide....16mo, mor., 2 50
Howard's Transition Curve Field-book..........................16mo, morocco, 1 50
Hudson's Tables for Calculating the Cubic Contents of Excavations and Embankments...8vo, 1 00
Molitor and Beard's Manual for Resident Engineers.......................16mo, 1 00
Nagle's Field Manual for Railroad Engineers....................16mo, morocco, 3 00
Philbrick's Field Manual for Engineers..........................16mo, morocco, 3 00
Searles's Field Engineering.....................................16mo, morocco, 3 00
Railroad Spiral...16mo, morocco, 1 50
Taylor's Prismoidal Formulæ and Earthwork...............................8vo, 1 50
* **Trautwine's Method ot Calculating the Cubic Contents of Excavations and Embankments by the Aid of Diagrams**.........................8vo, 2 00
The Field Practice of Laying Out Circular Curves for Railroads. 12mo, morocco, 2 50
Cross-section Sheet...Paper, 25
Webb's Railroad Construction. 2d Edition, Rewritten......16mo, morocco, 5 00
Wellington's Economic Theory of the Location of Railways......Small 8vo, 5 00

DRAWING.

Barr's Kinematics of Machinery...8vo, 2 50
* **Bartlett's Mechanical Drawing**..8vo, 3 00
* " Abridged Ed...8vo, 1 50
Coolidge's Manual of Drawing.......................................8vo, paper, 1 00
Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers...Oblong 4to. 2 50
Durley's Kinematics of Machines...8vo, 4 00

Hill's Text-book on Shades and Shadows, and Perspective...............8vo. 2 00
Jamison's Elements of Mechanical Drawing........................8vo, 2 50
Jones's Machine Design:
Part I.—Kinematics of Machinery...........................8vo, 1 50
Part II.—Form, Strength, and Proportions of Parts..............8vo, 3 00
MacCord's Elements of Descriptive Geometry.......................8vo, 3 00
Kinematics; or, Practical Mechanism........................8vo, 5 00
Mechanical Drawing.....................................4to, 4 00
Velocity Diagrams.......................................8vo, 1 50
Mahan's Descriptive Geometry and Stone-cutting..................8vo, 1 50
Industrial Drawing. (Thompson.)...........................8vo, 3 50
Moyer's Descriptive Geometry. (*In press.*)
Reed's Topographical Drawing and Sketching......................4to, 5 00
Reid's Course in Mechanical Drawing..........................8vo, 2 00
Text-book of Mechanical Drawing and Elementary Machine Design..8vo, 3 00
Robinson's Principles of Mechanism...........................8vo, 3 00
Schwamb and Merrill's Elements of Mechanism....................8vo, 3 00
Smith's Manual of Topographical Drawing. (McMillan.)..............8vo, 2 50
Warren's Elements of Plane and Solid Free-hand Geometrical Drawing..12mo, 1 00
Drafting Instruments and Operations.......................12mo, 1 25
Manual of Elementary Projection Drawing....................12mo, 1 50
Manual of Elementary Problems in the Linear Perspective of Form and Shadow.......................................12mo, 1 00
Plane Problems in Elementary Geometry......................12mo, 1 25
Primary Geometry......................................12mo, 75
Elements of Descriptive Geometry, Shadows, and Perspective......8vo, 3 50
General Problems of Shades and Shadows.......................8vo 3 00
Elements of Machine Construction and Drawing..................8vo, 7 50
Problems, Theorems, and Examples in Descriptive Geometry.......8vo, 2 50
Weisbach's Kinematics and the Power of Transmission. (Hermann and Klein.)..8vo, 5 00
Whelpley's Practical Instruction in the Art of Letter Engraving.......12mo, 2 00
Wilson's (H. M.) Topographic Surveying........................8vo, 3 50
Wilson's (V. T.) Free-hand Perspective..........................8vo, 2 50
Wilson's (V. T.) Free-hand Lettering..........................8vo, 1 00
Woolf's Elementary Course in Descriptive Geometry............Large 8vo, 3 00

ELECTRICITY AND PHYSICS.

Anthony and Brackett's Text-book of Physics. (Magie.)........Small 8vo, 3 00
Anthony's Lecture-notes on the Theory of Electrical Measurements....12mo, 1 00
Benjamin's History of Electricity..............................8vo, 3 00
Voltaic Cell..8vo, 3 00
Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.)..8vo, 3 00
Crehore and Squier's Polarizing Photo-chronograph..................8vo, 3 00
Dawson's "Engineering" and Electric Traction Pocket-book..16mo, morocco, 5 00
Dolezalek's Theory of the Lead Accumulator (Storage Battery). (Von Ende.)...12mo, 2 50
Duhem's Thermodynamics and Chemistry. (Burgess.)................8vo, 4 00
Flather's Dynamometers, and the Measurement of Power............12mo, 3 00
Gilbert's De Magnete. (Mottelay.)............................8vo, 2 50
Hanchett's Alternating Currents Explained......................12mo, 1 00
Hering's Ready Reference Tables (Conversion Factors)......16mo, morocco, 2 50
Holman's Precision of Measurements...........................8vo, 2 00
Telescopic Mirror-scale Method, Adjustments, and Tests.....Large 8vo, 75
Landauer's Spectrum Analysis. (Tingle.)........................8vo, 3 00
Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.)12mo 3 00
Löb's Electrolysis and Electrosynthesis of Organic Compounds. (Lorenz.) 12mo, 1 00

* Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II. 8vo, each, 6 00
* Michie. Elements of Wave Motion Relating to Sound and Light.......8vo, 4 00
Niaudet's Elementary Treatise on Electric Batteries. (Fishback.)......12mo, 2 50
* Rosenberg's Electrical Engineering. (Haldane Gee—Kinzbrunner.)....8vo, 1 50
Ryan, Norris, and Hoxie's Electrical Machinery. Vol. I................8vo, 2 50
Thurston's Stationary Steam-engines.............................8vo, 2 50
* Tillman's Elementary Lessons in Heat...............................8vo, 1 50
Tory and Pitcher's Manual of Laboratory Physics..............Small 8vo, 2 00
Ulke's Modern Electrolytic Copper Refining.........................8vo, 3 00

LAW.

* Davis's Elements of Law..8vo, 2 50
* Treatise on the Military Law of United States.....................8vo, 7 00
* Sheep, 7 50
Manual for Courts-martial..............................16mo, morocco, 1 50
Wait's Engineering and Architectural Jurisprudence...................8vo, 6 00
Sheep, 6 50
Law of Operations Preliminary to Construction in Engineering and Architecture.. 8vo, 5 00
Sheep, 5 50
Law of Contracts..8vo, 3 00
Winthrop's Abridgment of Military Law...........................12mo, 2 50

MANUFACTURES.

Bernadou's Smokeless Powder—Nitro-cellulose and Theory of the Cellulose Molecule..12mo, 2 50
Bolland's Iron Founder..12mo, 2 50
"The Iron Founder," Supplement.....................................12mo, 2 50
Encyclopedia of Founding and Dictionary of Foundry Terms Used in the Practice of Moulding...12mo, 3 00
Eissler's Modern High Explosives....................................8vo, 4 00
Effront's Enzymes and their Applications. (Prescott.)..................8vo 3 00
Fitzgerald's Boston Machinist.......................................18mo, 1 00
Ford's Boiler Making for Boiler Makers.............................18mo, 1 00
Hopkins's Oil-chemists' Handbook....................................8vo, 3 00
Keep's Cast Iron..8vo, 2 50
Leach's The Inspection and Analysis of Food with Special Reference to State Control. (*In preparation.*)
Matthews's The Textile Fibres.......................................8vo, 3 50
Metcalf's Steel. A Manual for Steel-users...........................12mo, 2 00
Metcalfe's Cost of Manufactures—And the Administration of Workshops, Public and Private...8vo, 5 00
Meyer's Modern Locomotive Construction..............................4to, 10 00
Morse's Calculations used in Cane-sugar Factories..........16mo, morocco, 1 50
* Reisig's Guide to Piece-dyeing......................................8vo, 25 00
Sabin's Industrial and Artistic Technology of Paints and Varnish 8vo, 3 00
Smith's Press-working of Metals.....................................8vo, 3 00
Spalding's Hydraulic Cement..12mo, 2 00
Spencer's Handbook for Chemists of Beet-sugar Houses.....16mo, morocco, 3 00
Handbook for Sugar Manufacturers and their Chemists...16mo morocco, 2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. (*In press.*)
Thurston's Manual of Steam-boilers, their Designs, Construction and Operation..8vo, 5 00
* Walke's Lectures on Explosives.....................................8vo, 4 00
West's American Foundry Practice...................................12mo, 2 50
Moulder's Text-book...12mo, 2 50

Wolff's Windmill as a Prime Mover 8vo, 3 00
Woodbury's Fire Protection of Mills 8vo, 2 50
Wood's Rustless Coatings: Corrosion and Electrolysis of Iron and Steel...8vo, 4 00

MATHEMATICS.

Baker's Elliptic Functions 8vo, 1 50
* **Bass's Elements of Differential Calculus** 12mo, 4 00
Briggs's Elements of Plane Analytic Geometry 12mo, 1 00
Compton's Manual of Logarithmic Computations 12mo, 1 50
Davis's Introduction to the Logic of Algebra 8vo, 1 50
* **Dickson's College Algebra** ....Large 12mo, 1 50
* Answers to Dickson's College Algebra ....8vo, paper, 25
* Introduction to the Theory of Algebraic Equations ....Large 12mo, 1 25
Halsted's Elements of Geometry 8vo, 1 75
Elementary Synthetic Geometry 8vo, 1 50
Rational Geometry 12mo,
* Johnson's (J. B.) Three-place Logarithmic Tables: Vest-pocket size..paper, 15
100 copies for 5 00
* Mounted on heavy cardboard, 8 × 10 inches, 25
10 copies for 2 00
Johnson's (W. W.) Elementary Treatise on Differential Calculus...Small 8vo, 3 00
Johnson's (W. W.) Elementary Treatise on the Integral Calculus. .Small 8vo, 1 50
Johnson's (W. W.) Curve Tracing in Cartesian Co-ordinates 12mo, 1 00
Johnson's (W. W.) Treatise on Ordinary and Partial Differential Equations.
Small 8vo, 3 50
Johnson's (W. W.) Theory of Errors and the Method of Least Squares..12mo, 1 50
* Johnson's (W. W.) Theoretical Mechanics 12mo, 3 00
Laplace's Philosophical Essay on Probabilities. (Truscott and Emory.) 12mo, 2 00
* **Ludlow and Bass. Elements of Trigonometry and Logarithmic and Other Tables** 8vo, 3 00
Trigonometry and Tables published separately................. Each, 2 00
* **Ludlow's Logarithmic and Trigonometric Tables** 8vo, 1 00
Maurer's Technical Mechanics 8vo, 4 00
Merriman and Woodward's Higher Mathematics 8vo, 5 00
Merriman's Method of Least Squares 8vo, 2 00
Rice and Johnson's Elementary Treatise on the Differential Calculus.Sm., 8vo, 3 00
Differential and Integral Calculus. 2 vols. in one..........Small 8vo, 2 50
Wood's Elements of Co-ordinate Geometry 8vo, 2 00
Trigonometry: Analytical, Plane, and Spherical..................12mo, 1 00

MECHANICAL ENGINEERING.

MATERIALS OF ENGINEERING, STEAM-ENGINES AND BOILERS.

Bacon's Forge Practice 12mo, 1 50
Baldwin's Steam Heating for Buildings 12mo, 2 50
Barr's Kinematics of Machinery 8vo, 2 50
* **Bartlett's Mechanical Drawing** 8vo, 3 00
* " " " Abridged Ed..................................8vo, 1 50
Benjamin's Wrinkles and Recipes 12mo, 2 00
Carpenter's Experimental Engineering 8vo, 6 00
Heating and Ventilating Buildings 8vo, 4 00
Cary's Smoke Suppression in Plants using Bituminous Coal. (*In preparation.*)
Clerk's Gas and Oil Engine ....Small 8vo, 4 00
Coolidge's Manual of Drawing ....8vo, paper, 1 00
Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers ....Oblong 4to, 2 50

Cromwell's Treatise on Toothed Gearing.........................12mo, 1 50
Treatise on Belts and Pulleys.........................12mo, 1 50
Durley's Kinematics of Machines.........................8vo, 4 00
Flather's Dynamometers and the Measurement of Power.............12mo, 3 00
Rope Driving.........................12mo, 2 00
Gill's Gas and Fuel Analysis for Engineers.........................12mo, 1 25
Hall's Car Lubrication.........................12mo, 1 00
Hering's Ready Reference Tables (Conversion Factors)......16mo, morocco, 2 50
Hutton's The Gas Engine.........................8vo, 5 00
Jamison's Mechanical Drawing.........................8vo, 2 50
Jones's Machine Design:
Part I.—Kinematics of Machinery.........................8vo, 1 50
Part II.—Form, Strength, and Proportions of Parts................8vo, 3 00
Kent's Mechanical Engineer's Pocket-book.................16mo, morocco, 5 00
Kerr's Power and Power Transmission.........................8vo, 2 00
Leonard's Machine Shops, Tools, and Methods. (*In press.*)
MacCord's Kinematics; or, Practical Mechanism.........................8vo, 5 00
Mechanical Drawing.........................4to, 4 00
Velocity Diagrams.........................8vo, 1 50
Mahan's Industrial Drawing. (Thompson.).........................8vo, 3 50
Poole's Calorific Power of Fuels.........................8vo, 3 00
Reid's Course in Mechanical Drawing.........................8vo, 2 00
Text-book of Mechanical Drawing and Elementary Machine Design..8vo, 3 00
Richards's Compressed Air.........................12mo, 1 50
Robinson's Principles of Mechanism.........................8vo, 3 00
Schwamb and Merrill's Elements of Mechanism.........................8vo, 3 00
Smith's Press-working of Metals.........................8vo, 3 00
Thurston's Treatise on Friction and Lost Work in Machinery and Mill Work.........................8vo, 3 00
Animal as a Machine and Prime Motor, and the Laws of Energetics.12mo, 1 00
Warren's Elements of Machine Construction and Drawing..............8vo, 7 50
Weisbach's Kinematics and the Power of Transmission. Herrmann—Klein.).........................8vo, 5 00
Machinery of Transmission and Governors. (Herrmann—Klein.)..8vo, 5 00
Hydraulics and Hydraulic Motors. (Du Bois.)....................8vo, 5 00
Wolff's Windmill as a Prime Mover.........................8vo, 3 00
Wood's Turbines.........................8vo, 2 50

MATERIALS OF ENGINEERING.

Bovey's Strength of Materials and Theory of Structures................8vo, 7 50
Burr's Elasticity and Resistance of the Materials of Engineering. 6th Edition Reset.........................8vo, 7 50
Church's Mechanics of Engineering.........................8vo, 6 00
Johnson's Materials of Construction.........................Large 8vo, 6 00
Keep's Cast Iron.........................8vo, 2 50
Lanza's Applied Mechanics.........................8vo, 7 50
Martens's Handbook on Testing Materials. (Henning.)...............8vo, 7 50
Merriman's Text-book on the Mechanics of Materials.................8vo, 4 00
Strength of Materials.........................12mo, 1 00
Metcalf's Steel. A Manual for Steel-users.........................12mo 2 00
Sabin's Industrial and Artistic Technology of Paints and Varnish......8vo, 3 00
Smith's Materials of Machines.........................12mo, 1 00
Thurston's Materials of Engineering.........................3 vols., 8vo, 8 00
Part II.—Iron and Steel.........................8vo, 3 50
Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents.........................8vo 2 50
Text-book of the Materials of Construction.........................8vo, 5 00

Wood's (De V.) Treatise on the Resistance of Materials and an Appendix on the Preservation of Timber. 8vo, 2 00
Wood's (De V.) Elements of Analytical Mechanics. 8vo, 3 00
Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel. 8vo, 4 00

STEAM-ENGINES AND BOILERS.

Carnot's Reflections on the Motive Power of Heat. (Thurston.). 12mo, 1 50
Dawson's "Engineering" and Electric Traction Pocket-book. . 16mo, mor., 5 00
Ford's Boiler Making for Boiler Makers. 18mo, 1 00
Goss's Locomotive Sparks. 8vo, 2 00
Hemenway's Indicator Practice and Steam-engine Economy. 12mo, 2 00
Hutton's Mechanical Engineering of Power Plants. 8vo, 5 00
Heat and Heat-engines. 8vo, 5 00
Kent's Steam-boiler Economy. 8vo, 4 00
Kneass's Practice and Theory of the Injector. 8vo, 1 50
MacCord's Slide-valves. 8vo, 2 00
Meyer's Modern Locomotive Construction. 4to, 10 00
Peabody's Manual of the Steam-engine Indicator. 12mo, 1 50
Tables of the Properties of Saturated Steam and Other Vapors. 8vo, 1 00
Thermodynamics of the Steam-engine and Other Heat-engines. 8vo, 5 00
Valve-gears for Steam-engines. 8vo, 2 50
Peabody and Miller's Steam-boilers. 8vo, 4 00
Pray's Twenty Years with the Indicator. . . . . Large 8vo, 2 50
Pupin's Thermodynamics of Reversible Cycles in Gases and Saturated Vapors. (Osterberg.). 12mo, 1 25
Reagan's Locomotives: Simple, Compound, and Electric. 12mo, 2 50
Rontgen's Principles of Thermodynamics. (Du Bois.). 8vo, 5 00
Sinclair's Locomotive Engine Running and Management. 12mo, 2 00
Smart's Handbook of Engineering Laboratory Practice. 12mo, 2 50
Snow's Steam-boiler Practice. 8vo, 3 00
Spangler's Valve-gears. 8vo, 2 50
Notes on Thermodynamics. 12mo, 1 00
Spangler, Greene, and Marshall's Elements of Steam-engineering. 8vo, 3 00
Thurston's Handy Tables. 8vo, 1 50
Manual of the Steam-engine. . . . . 2 vols.. 8vo, 10 00
Part I.—History, Structuce, and Theory. 8vo, 6 00
Part II.—Design, Construction, and Operation. 8vo, 6 00
Handbook of Engine and Boiler Trials, and the Use of the Indicator and the Prony Brake. 8vo, 5 00
Stationary Steam-engines. 8vo, 2 50
Steam-boiler Explosions in Theory and in Practice. 12mo, 1 50
Manual of Steam-boilers, Their Designs, Construction, and Operation. 8vo, 5 00
Weisbach's Heat, Steam, and Steam-engines. (Du Bois.). 8vo, 5 00
Whitham's Steam-engine Design. 8vo, 5 00
Wilson's Treatise on Steam-boilers. (Flather.). 16mo, 2 50
Wood's Thermodynamics Heat Motors, and Refrigerating Machines. . . . 8vo, 4 00

MECHANICS AND MACHINERY.

Barr's Kinematics of Machinery. 8vo, 2 50
Bovey's Strength of Materials and Theory of Structures. 8vo, 7 50
Chase's The Art of Pattern-making. 12mo, 2 50
Chordal.—Extracts from Letters. 12mo, 2 00
Church's Mechanics of Engineering. 8vo, 6 00

Church's Notes and Examples in Mechanics..........................8vo, 2 00
Compton's First Lessons in Metal-working..........................12mo, 1 50
Compton and De Groodt's The Speed Lathe..........................12mo, 1 50
Cromwell's Treatise on Toothed Gearing..........................12mo, 1 50
Treatise on Belts and Pulleys..........................12mo, 1 50
Dana's Text-book of Elementary Mechanics for the Use of Colleges and Schools..........................12mo, 1 50
Dingey's Machinery Pattern Making..........................12mo, 2 00
Dredge's Record of the Transportation Exhibits Building of the World's Columbian Exposition of 1893..........................4to half morocco, 5 00
Du Bois's Elementary Principles of Mechanics:
Vol. I.—Kinematics..........................8vo, 3 50
Vol. II.—Statics..........................8vo, 4 00
Vol. III.—Kinetics..........................8vo, 3 50
Mechanics of Engineering. Vol. I..........................Small 4to, 7 50
Vol. II..........................Small 4to, 10 00
Durley's Kinematics of Machines..........................8vo, 4 00
Fitzgerald's Boston Machinist..........................16mo, 1 00
Flather's Dynamometers, and the Measurement of Power..........................12mo, 3 00
Rope Driving..........................12mo, 2 00
Goss's Locomotive Sparks..........................8vo, 2 00
Hall's Car Lubrication..........................12mo, 1 00
Holly's Art of Saw Filing..........................18mo, 75
* Johnson's (W. W.) Theoretical Mechanics..........................12mo, 3 00
Johnson's (L. J.) Statics by Graphic and Algebraic Methods..........................8vo, 2 00
Jones's Machine Design:
Part I.—Kinematics of Machinery..........................8vo, 1 50
Part II.—Form, Strength, and Proportions of Parts..........................8vo, 3 00
Kerr's Power and Power Transmission..........................8vo, 2 00
Lanza's Applied Mechanics..........................8vo, 7 50
Leonard's Machine Shops, Tools, and Methods. (*In press.*)
MacCord's Kinematics; or, Practical Mechanism..........................8vo, 5 00
Velocity Diagrams..........................8vo, 1 50
Maurer's Technical Mechanics..........................8vo, 4 00
Merriman's Text-book on the Mechanics of Materials..........................8vo, 4 00
* Michie's Elements of Analytical Mechanics..........................8vo, 4 00
Reagan's Locomotives: Simple, Compound, and Electric..........................12mo, 2 50
Reid's Course in Mechanical Drawing..........................8vo, 2 00
Text-book of Mechanical Drawing and Elementary Machine Design..8vo, 3 00
Richards's Compressed Air..........................12mo, 1 50
Robinson's Principles of Mechanism..........................8vo, 3 00
Ryan, Norris, and Hoxie's Electrical Machinery. Vol. I..........................8vo, 2 50
Schwamb and Merrill's Elements of Mechanism..........................8vo, 3 00
Sinclair's Locomotive-engine Running and Management..........................12mo, 2 00
Smith's Press-working of Metals..........................8vo, 3 00
Materials of Machines..........................12mo, 1 00
Spangler, Greene, and Marshall's Elements of Steam-engineering..........................8vo, 3 00
Thurston's Treatise on Friction and Lost Work in Machinery and Mill Work..........................8vo, 3 00
Animal as a Machine and Prime Motor, and the Laws of Energetics.12mo, 1 00
Warren's Elements of Machine Construction and Drawing..........................8vo, 7 50
Weisbach's Kinematics and the Power of Transmission. (Herrmann—Klein.)..........................8vo, 5 00
Machinery of Transmission and Governors. (Herrmann—Klein.).8vo, 5 00
Wood's Elements of Analytical Mechanics..........................8vo, 3 00
Principles of Elementary Mechanics..........................12mo, 1 25
Turbines..........................8vo, 2 50
The World's Columbian Exposition of 1893..........................4to, 1 00

METALLURGY.

Egleston's Metallurgy of Silver, Gold, and Mercury:

Vol. I.—Silver 8vo, 7 50

Vol. II.—Gold and Mercury 8vo, 7 50

** Iles's Lead-smelting. (Postage 9 cents additional.) 12mo, 2 50

Keep's Cast Iron 8vo, 2 50

Kunhardt's Practice of Ore Dressing in Europe 8vo, 1 50

Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.) . 12mo, 3 00

Metcalf's Steel. A Manual for Steel-users 12mo, 2 00

Smith's Materials of Machines 12mo, 1 00

Thurston's Materials of Engineering. In Three Parts 8vo, 8 00

Part II.—Iron and Steel 8vo, 3 50

Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents 8vo, 2 50

Ulke's Modern Electrolytic Copper Refining 8vo, 3 00

MINERALOGY.

Barringer's Description of Minerals of Commercial Value. Oblong, morocco, 2 50

Boyd's Resources of Southwest Virginia 8vo. 3 00

Map of Southwest Virginia .... Pocket-book form, 2 00

Brush's Manual of Determinative Mineralogy. (Penfield.) 8vo, 4 00

Chester's Catalogue of Minerals .... 8vo, paper, 1 00

Cloth, 1 25

Dictionary of the Names of Minerals 8vo, 3 50

Dana's System of Mineralogy .... Large 8vo, half leather, 12 50

First Appendix to Dana's New "System of Mineralogy." .... Large 8vo, 1 00

Text-book of Mineralogy 8vo, 4 00

Minerals and How to Study Them 12mo, 1 50

Catalogue of American Localities of Minerals .... Large 8vo, 1 00

Manual of Mineralogy and Petrography 12mo, 2 00

Douglas's Untechnical Addresses on Technical Subjects 12mo, 1 00

Eakle's Mineral Tables 8vo, 1 25

Egleston's Catalogue of Minerals and Synonyms 8vo, 2 50

Hussak's The Determination of Rock-forming Minerals. (Smith.) Small 8vo, 2 00

Merrill's Non-metallic Minerals: Their Occurrence and Uses 8vo, 4 00

* Penfield's Notes on Determinative Mineralogy and Record of Mineral Tests. 8vo, paper, 0 50

Rosenbusch's Microscopical Physiography of the Rock-making Minerals. (Iddings.) 8vo, 5 00

* Tillman's Text-book of Important Minerals and Docks 8vo, 2 00

Williams's Manual of Lithology 8vo, 3 00

MINING.

Beard's Ventilation of Mines 12mo, 2 50

Boyd's Resources of Southwest Virginia 8vo, 3 00

Map of Southwest Virginia .... Pocket-book form, 2 00

Douglas's Untechnical Addresses on Technical Subjects 12mo, 1 00

* Drinker's Tunneling, Explosive Compounds, and Rock Drills. 4to, half morocco, 25 00

Eissler's Modern High Explosives 8vo, 4 00

Fowler's Sewage Works Analyses 12mo, 2 00

Goodyear's Coal-mines of the Western Coast of the United States 12mo, 2 50

Ihlseng's Manual of Mining 8vo, 4 00

** Iles's Lead-smelting. (Postage 9c. additional.) 12mo, 2 50

Kunhardt's Practice of Ore Dressing in Europe 8vo, 1 50

O'Driscoll's Notes on the Treatment of Gold Ores 8vo, 2 00

* Walke's Lectures on Explosives 8vo, 4 00

Wilson's Cyanide Processes 12mo, 1 50

Chlorination Process 12mo, 1 50

Wilson's Hydraulic and Placer Mining............................12mo, 2 00
Treatise on Practical and Theoretical Mine Ventilation...........12mo, 1 25

SANITARY SCIENCE.

Folwell's Sewerage. (Designing, Construction, and Maintenance.)......8vo, 3 00
Water-supply Engineering..................................8vo, 4 00
Fuertes's Water and Public Health..............................12mo, 1 50
Water-filtration Works..12mo, 2 50
Gerhard's Guide to Sanitary House-inspection.....................16mo, 1 00
Goodrich's Economical Disposal of Town's Refuse...............Demy 8vo, 3 50
Hazen's Filtration of Public Water-supplies..........................8vo, 3 00
Leach's The Inspection and Analysis of Food with Special Reference to State Control..8vo, 7 50
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten..............................8vo, 4 00
Examination of Water. (Chemical and Bacteriological.)........12mo, 1 25
Merriman's Elements of Sanitary Engineering.......................8vo, 2 00
Ogden's Sewer Design...12mo, 2 00
Prescott and Winslow's Elements of Water Bacteriology, with Special Reference to Sanitary Water Analysis...............................12mo, 1 25
* Price's Handbook on Sanitation 12mo, 1 50
Richards's Cost of Food. A Study in Dietaries.....................12mo, 1 00
Cost of Living as Modified by Sanitary Science..................12mo, 1 00
Richards and Woodman's Air, Water, and Food from a Sanitary Standpoint..8vo, 2 00
* Richards and Williams's The Dietary Computer....................8vo, 1 50
Rideal's Sewage and Bacterial Purification of Sewage.................8vo, 3 50
Turneaure and Russell's Public Water-supplies........................8vo, 5 00
Von Behring's Suppression of Tuberculosis. (Bolduan.).............12mo, 1 00
Whipple's Microscopy of Drinking-water...........................8vo, 3 50
Woodhull's Notes and Military Hygiene............................16mo, 1 50

MISCELLANEOUS.

Emmons's Geological Guide-book of the Rocky Mountain Excursion of the International Congress of Geologists..................Large 8vo, 1 50
Ferrel's Popular Treatise on the Winds..............................8vo, 4 00
Haines's American Railway Management 12mo 2 50
Mott's Composition, Digestibility, and Nutritive Value of Food. Mounted chart. 1 25
Fallacy of the Present Theory of Sound.......................16mo, 1 00
Ricketts's History of Rensselaer Polytechnic Institute, 1824–1894. Small 8vo, 3 00
Rostoski's Serum Diagnosis. (Bolduan.)...........................12mo, 1 00
Rotherham's Emphasized New Testament......................Large 8vo, 2 00
Steel's Treatise on the Diseases of the Dog..........................8vo, 3 50
Totten's Important Question in Metrology...........................8vo, 2 50
The World's Columbian Exposition of 1893............................4to, 1 00
Von Behring's Suppression of Tuberculosis. (Bolduan.).............12mo, 1 00
Worcester and Atkinson. Small Hospitals, Establishment and Maintenance, and Suggestions for Hospital Architecture, with Plans for a Small Hospital..12mo, 1 25

HEBREW AND CHALDEE TEXT-BOOKS.

Green's Grammar of the Hebrew Language..........................8vo, 3 00
Elementary Hebrew Grammar...............................12mo, 1 25
Hebrew Chrestomathy...8vo, 2 00
Gesenius's Hebrew and Chaldee Lexicon to the Old Testament Scriptures. (Tregelles.).........................Small 4to, half morocco, 5 00
Letteris's Hebrew Bible..8vo, 2 25